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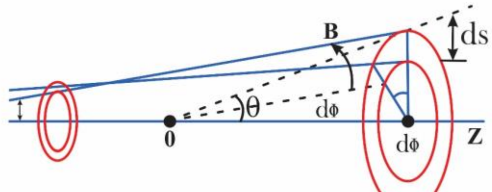

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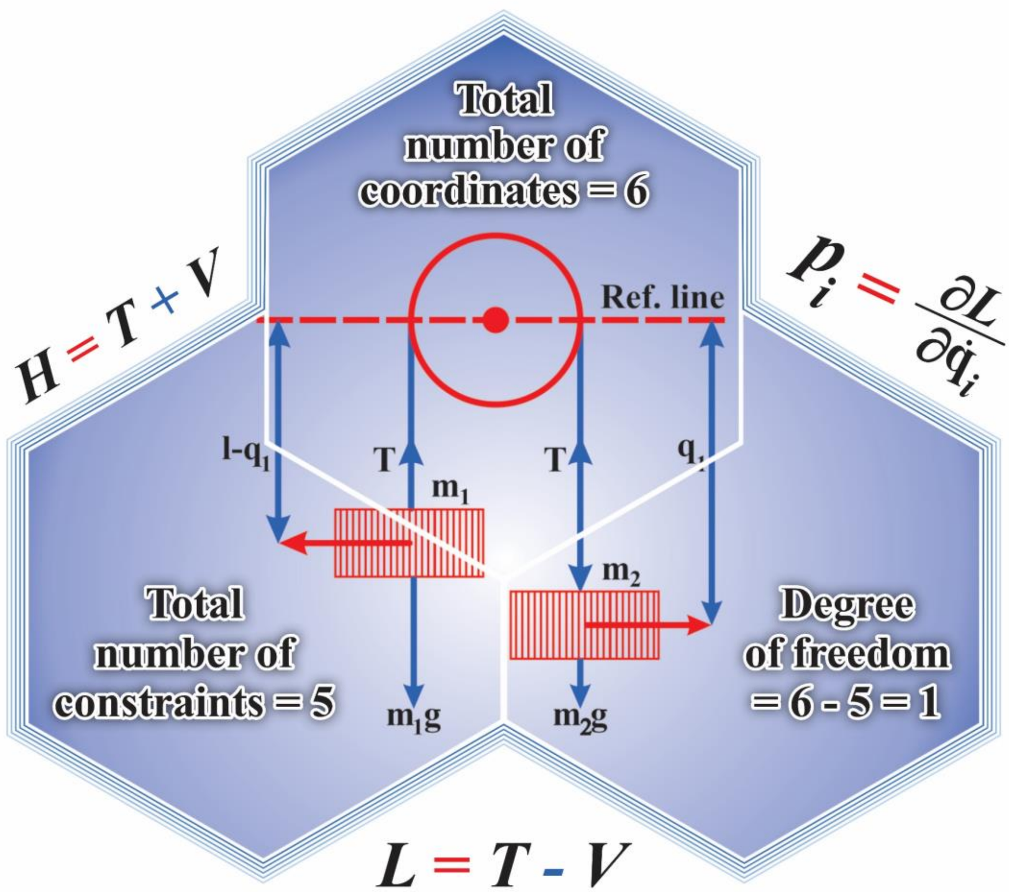
CLASSICAL MECHANICS

For BS/M.Sc Physics Programme

5th Edition



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TEACH YOURSELF

CLASSICAL MECHANICS

5th Edition

For **BS/M.Sc Physics** students of all Pakistani Universities

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Contents

Chapter 1

Elementary Particles

SOLVED PROBLEMS

Problem: 1.1- Derive Lagrange's equation of motion using Newton's laws.

Solution

In order to derive Lagrange's equation, consider case of a single particle. Relation between cartesian and generalized coordinates in one dimension is,

$$
\begin{aligned}
x_i =& x_i(q_1, q_2, q_3, \cdots\cdots, q_n, t) \\
\Rightarrow \quad dx_i =& \frac{\partial x_i}{\partial q_1}dq_1 + \frac{\partial x_i}{\partial q_2}dq_2 + \frac{\partial x_i}{\partial q_3}dq_3 + \cdots\cdots + \frac{\partial x_i}{\partial q_n}dq_n + \frac{\partial x_i}{\partial t}dt \\
\Rightarrow \quad dx_i =& \sum_j \frac{\partial x_i}{\partial q_j}dq_j + \frac{\partial x_i}{\partial t}dt \quad \Rightarrow \quad \frac{dx_i}{dt} = \sum_j \frac{\partial x_i}{\partial q_j}\frac{dq_j}{dt} + \frac{\partial x_i}{\partial t} \\
\Rightarrow \quad \dot{x}_i =& \sum_j \frac{\partial x_i}{\partial q_j}\dot{q}_j + \frac{\partial x_i}{\partial t} \quad \Rightarrow \quad \frac{\partial \dot{x}_i}{\partial \dot{q}_j} = \frac{\partial x_i}{\partial q_j}
\end{aligned}
$$

Generalized momentum is,

$$
\begin{aligned}
p_j =& \frac{\partial L}{\partial \dot{q}_j} = \frac{\partial}{\partial \dot{q}_j}(T - V) = \frac{\partial T}{\partial \dot{q}_j} = \frac{\partial}{\partial \dot{q}_j}\sum_i \frac{1}{2}m_i\dot{x}_i^2 = \sum_i m_i\dot{x}_i\frac{\partial \dot{x}_i}{\partial \dot{q}_j} = \sum_i m_i\dot{x}_i\frac{\partial x_i}{\partial q_j} \\
\Rightarrow \quad \dot{p}_j =& \sum_i m_i\left(\ddot{x}_i\frac{\partial x_i}{\partial q_j} + \dot{x}_i\frac{d}{dt}\frac{\partial x_i}{\partial q_j}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_i m_i \ddot{x}_i \frac{\partial x_i}{\partial q_j} + \sum_i m_i \left(\sum_k \frac{\partial^2 x_i}{\partial q_k \partial q_j} \dot{q}_k + \frac{\partial^2 x_i}{\partial q_j \partial t} \right) \\
\Rightarrow \qquad \frac{dp_j}{dt} &= \sum_i m_i \ddot{x}_i \frac{\partial x_i}{\partial q_j} + \sum_{i,k} m_i \dot{x}_i \frac{\partial^2 x_i}{\partial q_k \partial q_j} \dot{q}_k + \sum_{i,k} m_i \dot{x}_i \frac{\partial^2 x_i}{\partial t \partial q_j} \\
\Rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) &= Q_j + \sum_i m_i \dot{x}_i \frac{\partial}{\partial q_j} \left(\sum_k \frac{\partial x_i}{\partial q_k} \dot{q}_k + \frac{\partial x_i}{\partial t} \right)
\end{aligned}
$$

Since,

$$
\frac{\partial T}{\partial q_j} = \sum_i m_i \dot{x}_i \frac{\partial \dot{x}_i}{\partial q_j} = \sum_i m_i \dot{x}_i \frac{\partial}{\partial q_j} \left(\sum_k \frac{\partial x_i}{\partial q_k} \dot{q}_k + \frac{\partial x_i}{\partial t} \right)
$$

So,

$$
\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) = Q_j + \frac{\partial T}{\partial q_j} \qquad \Rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j
$$

For conservative forces,

$$
Q_j = -\frac{\partial V}{\partial q_j}
$$

So above equation becomes,

$$
\begin{aligned}
&\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = -\frac{\partial V}{\partial q_j} \quad \Rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} + \frac{\partial V}{\partial q_j} = 0 \\
&\Rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial}{\partial q_j}(T - V) = 0
\end{aligned}
$$

Now potential energy V is a function of position only, then it is independent of generalized velocities $\dot{q}_j$ and we can write;

$$
\frac{d}{dt}\left(\frac{\partial (T - V)}{\partial \dot{q}_j}\right) - \frac{\partial}{\partial q_j}(T - V) = 0
$$

In term of Lagrangian above equation becomes,

$$
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0, \qquad j = 1, 2, 3, \cdots\cdots, n
$$

Where n is the number of degree of freedom of system. These n second order differential equations are called Lagrange equations or D'Alembert form of Lagrange equations for a conservative, holonomic dynamical system

Problem: 1.2- Obtain Lagrangian and equation of motion for a double pendulum, where the lengths of pendula are l_1 and l_2 with corresponding masses m_1 and m_2.

Solution

Consider motion of the system in $x-y$ plane. Total number of coordinates are 4 and total number of constraints are 2 (1 for m_1 and 1 for m_2). So number of degrees of freedom $= 4-2=2$, $i.e.$ $j=1,2$. To obtain equation of motion, we first express K.E and P.E in terms of position co-ordinates;

For mass m_1;

$$\begin{aligned} x_1 =& l_1\cos\theta_1 \longrightarrow (i) \\ y_1 =& l_1\sin\theta_1 \longrightarrow (ii) \end{aligned}$$

For mass m_2;

$$\begin{aligned} x_2 =& l_1\cos\theta_1 + l_2\cos\theta_2 \longrightarrow (iii) \\ y_2 =& l_1\sin\theta_1 + l_2\sin\theta_2 \longrightarrow (iv) \end{aligned}$$

Thus differentiating w.r.t time t, we can write as;

$$\begin{aligned} \dot{x}_1 =& \frac{d}{dt}(l_1\cos\theta_1) = -l_1\dot{\theta}_1\sin\theta_1, \quad \dot{y}_1 = \frac{d}{dt}(l_1\sin\theta_1) = l_1\dot{\theta}_1\cos\theta_1 \\ \dot{x}_2 =& -l_1\dot{\theta}_1\sin\theta_1 - l_2\dot{\theta}_2\sin\theta_2, \qquad \dot{y}_2 = l_1\dot{\theta}_1\cos\theta_1 + l_2\dot{\theta}_2\cos\theta_2 \end{aligned}$$

The K.E of the system is;

$$\begin{aligned} T =& T_1 + T_2 = \frac{1}{2}m_1(\dot{x}_1^2+\dot{y}_1^2) + \frac{1}{2}m_2(\dot{x}_2^2+\dot{y}_2^2) \longrightarrow (v) \\ T =& \frac{1}{2}m_1(l_1^2\dot{\theta}_1^2\sin^2\theta_1 + l_1^2\dot{\theta}_1^2\cos^2\theta_1) + \frac{1}{2}m_2[(l_1^2\dot{\theta}_1^2\sin^2\theta_1 + l_2^2\dot{\theta}_2^2\cos^2\theta_1 \\ &+ 2l_1l_2\dot{\theta}_1\dot{\theta}_2\sin\theta_1\sin\theta_2) + (l_1^2\dot{\theta}_1^2\cos^2\theta_1 + l_2^2\dot{\theta}_2^2 + 2l_1l_2\dot{\theta}_1\dot{\theta}_2\cos\theta_1\cos\theta_2)] \end{aligned}$$

$$T = \frac{1}{2}m_1(l_1^2\dot{\theta}_1^2) + \frac{1}{2}m_2[l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1l_2\dot{\theta}_1\dot{\theta}_2(\sin\theta_1\sin\theta_2 + \cos\theta_1\cos\theta_2)]$$
$$T = \frac{1}{2}m_1(l_1^2\dot{\theta}_1^2) + \frac{1}{2}m_2l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2l_2^2\dot{\theta}_2^2 + m_2l_1l_2\dot{\theta}_1\dot{\theta}_2[\cos(\theta_1 - \theta_2)]$$

or,

$$T = \frac{1}{2}(m_1 - m_2)l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2l_2^2\dot{\theta}_2^2 + m_2l_1l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2) \longrightarrow (vi)$$

Now the potential energy is given by;

$$V = m_1gh_1 + m_2gh_2 \longrightarrow (vii)$$

For first pendulum $\therefore h_1 = l_1 + l_2 - x_1$

$$h_1 = (l_1 + l_2 - l_1\cos\theta_1) \longrightarrow (viii)$$

For second pendulum $\therefore h_2 = l_1 + l_2 - x_2$

$$h_2 = l_1 + l_2 - (l_1\cos\theta_1 + l_2\cos\theta_2) \longrightarrow (ix)$$

Therefore;

$$V = m_1g(l_1 + l_2 - l_1\cos\theta_1) + m_2g(l_1 + l_2 - l_1\cos\theta_1 - l_2\cos\theta_2) \longrightarrow (x)$$

The Lagrangian L is defined as;

$$L = T - V$$

or $$L = \frac{1}{2}l_1^2\dot{\theta}_1^2(m_1 + m_2) + \frac{1}{2}m_2l_2^2\dot{\theta}_2^2 + m_2l_1l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2) -$$
$$m_1g(l_1 + l_2 - l_1\cos\theta_1) - m_2g(l_1 + l_2 - \cos\theta_1 - l_2\cos\theta_2) \longrightarrow (xi)$$

Now, Lagrange's equation of motion are;

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}_1}\right) - \left(\frac{\partial L}{\partial\theta_1}\right) = 0 \rightarrow (1), \quad \text{as for } \; j = 1, \dot{q}_j = \dot{q}_1 = \dot{\theta}_1, \; \textit{and} \; q_j = q_1 = \theta_1$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}_2}\right) - \left(\frac{\partial L}{\partial\theta_2}\right) = 0 \rightarrow (2), \quad \text{as for } \; j = 2, \dot{q}_j = \dot{q}_2 = \dot{\theta}_2, \; \textit{and} \; q_j = q_2 = \theta_2$$

Now, differentiating partially Eq,(xi) w.r.t θ_1 and $\dot{\theta}_1$ we get

$$\frac{\partial L}{\partial \theta_1} = -m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - m_1 g l_1 \sin\theta_1 - m_2 g l_1 \sin\theta_1 \longrightarrow (3)$$

$$\frac{\partial L}{\partial \dot{\theta}_1} = l_1^2 \dot{\theta}_1 (m_1 + m_2) + m_2 l_1 l_2 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \longrightarrow (4)$$

Now

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) = l_1^2 \ddot{\theta}_1 (m_1 + m_2) + m_2 l_1 l_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) - m_2 l_1 l_2 \dot{\theta}_2 (\dot{\theta}_1 - \dot{\theta}_2) \sin(\theta_1 - \theta_2) \longrightarrow (5)$$

Put equations (3) and (5) into Eq. (1)

$$l_1^2 \ddot{\theta}_1 (m_1 + m_2) + m_2 l_1 l_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) - m_2 l_1 l_2 \dot{\theta}_2 \sin(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2)$$
$$+ m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) + m_1 g l_1 \sin\theta_1 + m_2 g l_1 \sin\theta_1 = 0$$

or

$$l_1^2 \ddot{\theta}_1 (m_1 + m_2) + m_2 l_1 l_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) - m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2)$$
$$+ m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) + m_2 l_1 l_2 \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) + (m_1 + m_2) g l_1 \sin\theta_1 = 0$$
$$(m_1 + m_2) l_1^2 \ddot{\theta}_1 + m_2 l_1 l_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) + m_2 l_1 l_2 \dot{\theta}_2^2 \sin(\theta_1 - \theta_2)$$
$$= -(m_1 + m_2) g l_1 \sin\theta_1$$

This is the result of Eq.(1). Similarly the result of Eq.(2) is

$$(m_1 + m_2) l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) + m_2 l_1 l_2 \dot{\theta}_1^2 \sin(\theta_1 - \theta_2)$$
$$= -m_2 g l_2 \sin\theta_2$$

Problem: 1.3- The magnitude of force of attraction between positively charged proton and negatively charged electron in hydrogen atom is,

$$F = k\frac{e^2}{r^2}$$

Show that change in kinetic energy of electron is,

$$\frac{1}{2} k e^2 \left(\frac{1}{r_2} - \frac{1}{r_1}\right)$$

Where $r_2 > r_1$ being radii of two circular orbits.

By how much has the total energy of atom changed is this process?

Solution

As the electron is revolving in circular orbit, so given force provides necessary centripetal force i.e.,

$$\frac{mv^2}{r} = k\frac{e^2}{r^2} \qquad \Rightarrow \frac{1}{2}mv^2 = k\frac{e^2}{2r} \qquad \Rightarrow K = k\frac{e^2}{2r}$$

Kinetic energy of orbit of radius r_1 is,

$$K_1 = k\frac{e^2}{2r_1}$$

Kinetic energy of orbit of radius r_2 is,

$$K_2 = k\frac{e^2}{2r_2}$$

Change in kinetic energy is,

$$\mathit{\Delta} K = K_2 - K_1 = K_2 = k\frac{e^2}{2r_2} - K_1 = k\frac{e^2}{2r_1} \qquad \Rightarrow \mathit{\Delta} K = \frac{1}{2}ke^2\left(\frac{1}{r_2} - \frac{1}{r_1}\right)$$

Change in total energy is,

$$\begin{aligned} \mathit{\Delta} E =& \mathit{\Delta} K + \mathit{\Delta} U = \frac{1}{2}ke^2\left(\frac{1}{r_2} - \frac{1}{r_1}\right) + \int_{r_2}^{r_1} -\frac{ke^2}{r^2}dr \\ =& \frac{1}{2}ke^2\left(\frac{1}{r_2} - \frac{1}{r_1}\right) - ke^2\left|-\frac{1}{r}\right|_{r_2}^{r_1} \\ \Rightarrow \qquad \mathit{\Delta} E =& \frac{1}{2}ke^2\left(\frac{1}{r_2} - \frac{1}{r_1}\right) - ke^2\left(\frac{1}{r_2} - \frac{1}{r_1}\right) = -\frac{1}{2}ke^2\left(\frac{1}{r_2} - \frac{1}{r_1}\right) \end{aligned}$$

This is required change in total energy.

Problem: 1.4- A Lagrangian for a particular physical system can be written as,

$$L' = \frac{m}{2}(a\dot{x}^2 + 2b\dot{x}\dot{y} + c\dot{y}^2) - \frac{k}{2}(ax^2 + 2bxy + cy^2)$$

Where a, b and c are arbitrary constants but subject to the condition that $b^2 - ac \neq 0$. What are the equations of motion? Examine particularly the two cases $a = 0 = c$ and

$b = 0, \quad c = -a$. What is the physical system described by the above Lagrangian? What is the significance of the condition on the value of $b^2 - ac$?

Solution

There are two degrees of freedom, i.e. x and y so that $j = 1, 2$

$$
\begin{aligned}
L' &= \frac{m}{2}(a\dot{x}^2 + 2b\dot{x}\dot{y} + c\dot{y}^2) - \frac{k}{2}(ax^2 + 2bxy + cy^2) \longrightarrow (1) \\
\frac{\partial L'}{\partial x} &= 0 - \frac{k}{2}(2ax + 2by + 0) = -k(ax + by) \longrightarrow (2) \\
\frac{\partial L'}{\partial \dot{x}} &= \frac{m}{2}(2a\dot{x} + 2b\dot{y} + 0) - 0 = m(a\dot{x} + b\dot{y}) \longrightarrow (3)
\end{aligned}
$$

The Lagrange's equation for $L = L'$ will become;

$$
\begin{aligned}
&\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{x}}\right) - \frac{\partial L'}{\partial x} = 0 \to (4), \quad \text{as for } \; j = 1, \quad \dot{q}_j = \dot{q}_1 = \dot{x}, \quad \textit{and} \quad q_j = q_1 = x \\
&\Rightarrow \quad \frac{d}{dt}[m(a\dot{x} + b\dot{y})] - [-k(ax - by)] = 0 \\
&\Rightarrow \quad m(a\ddot{x} + b\ddot{y}) = -k(ax + by) \longrightarrow (5)
\end{aligned}
$$

The Lagrange's equation for $L = L'$ will become;

$$
\begin{aligned}
&\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{y}}\right) - \frac{\partial L'}{\partial y} = 0 \to (6), \quad \text{as for } \; j = 2, \quad \dot{q}_j = \dot{q}_2 = \dot{y}, \quad \textit{and} \quad q_j = q_2 = y \\
&\frac{\partial L'}{\partial y} = 0 - \frac{k}{2}(0 + 2bx + 2cy) = -k(bx + cy) \longrightarrow (7) \\
&\frac{\partial L'}{\partial \dot{y}} = \frac{m}{2}(0 + 2b\dot{x} + 2c\dot{y}) - 0 = m(b\dot{x} + c\dot{y}) \longrightarrow (8)
\end{aligned}
$$

Similarly for y, substituting values in Eq. (6), we obtain

$$
\begin{aligned}
m(b\ddot{x} + c\ddot{y}) + k(bx + cy) &= 0 \\
m(b\ddot{x} + c\ddot{y}) &= -k(bx + cy) \longrightarrow (9)
\end{aligned}
$$

These are the equations of motion for a particle of mass m undergoing simple harmonic motion in two dimensions, as if bound by two springs of spring constant k.

$$\begin{aligned} \text{Let} \qquad u_1 =& ax + by \quad \Rightarrow \dot{u}_1 = a\dot{x} + b\dot{y} \quad \Rightarrow \ddot{u}_1 = a\ddot{x} + b\ddot{y} \\ \& \qquad u_2 =& bx + cy \quad \Rightarrow \dot{u}_2 = b\dot{x} + c\dot{y} \quad \Rightarrow \ddot{u}_2 = b\ddot{x} + c\ddot{y} \end{aligned}$$

So that Eqs.(5) and (9) can be written as;

$$\begin{aligned} m\ddot{u}_1 =& - ku_1 \longrightarrow (10) \\ \text{and} \quad m\ddot{u}_2 =& - ku_2 \longrightarrow (11) \end{aligned}$$

Now for case I, $a = 0, \quad c = 0$, so from equations (5) and (9) we have

$$\begin{aligned} & mb\ddot{y} = -kby \quad or \quad m\ddot{y} = -ky \quad or \quad \ddot{y} = -\frac{k}{m}y \longrightarrow (12) \\ \text{and} \qquad & mb\ddot{x} = -kbx \quad or \quad m\ddot{x} = -kx \quad or \quad \ddot{x} = -\frac{k}{m}x \longrightarrow (13) \end{aligned}$$

For case II, when $b = 0$ and $c = -a$, equation (5) and (9) will become

$$\begin{aligned} & ma\ddot{x} = -kax \qquad \Rightarrow \quad m\ddot{x} = -kx \quad or \ \ \ddot{x} = -\frac{k}{m}x \longrightarrow (14) \\ \text{and} \qquad & - ma\ddot{y} = -k(-ay) \quad \Rightarrow \quad m\ddot{y} = -ky \ \ or \ \ \ddot{y} = -\frac{k}{m}y \longrightarrow (15) \end{aligned}$$

In both cases, we have one dimensional harmonic oscillator.
The condition $b^2 - ac \neq 0$ is the condition that the coordinates transformation cannot be degenerate, i.e., there are actually two distinct dimensions in which the particle experiences a restoring force. If we have $b^2 = ac$, then we have just a one-dimensional problem.

Problem: 1.5- Consider the motion of a particle of mass m moving in space. Selecting the cylindrical coordinates (r, f, z) as the generalized coordinates, calculate the generalized force components if a force F acts on it.

Solution

The generalized force corresponding to the coordinate q_j

$$Q_j = F_i \cdot \frac{\partial r_i}{\partial q_j} = F_x \frac{\partial x}{\partial q_j} + F_y \frac{\partial y}{\partial q_j} + F_z \frac{\partial z}{\partial q_j}$$

In cylindrical co-ordinates

$$
\begin{aligned}
x =& \rho\cos\phi \quad y = \rho\sin\phi \quad and \quad z = z \\
\frac{\partial x}{\partial \rho} =& \cos\phi \quad \frac{\partial x}{\partial \phi} = -\rho\sin\phi \quad \frac{\partial x}{\partial z} = 0 \\
\frac{\partial y}{\partial \rho} =& \sin\phi \quad \frac{\partial y}{\partial \phi} = \rho\cos\phi \quad \frac{\partial y}{\partial z} = 0 \\
\frac{\partial z}{\partial \rho} =& 0 \quad \frac{\partial z}{\partial \phi} = 0 \quad \frac{\partial z}{\partial z} = 1
\end{aligned}
$$

Substituting these values in the expression for generalized force, we have

$$
\begin{aligned}
Q_\rho =& F_x\frac{\partial x}{\partial \rho} + F_y\frac{\partial y}{\partial \rho} + F_z\frac{\partial z}{\partial \rho} \\
=& F_x\cos\phi + F_y\sin\phi = F_\rho \\
Q_\phi =& -F_x\rho\sin\phi + F_y\rho\cos\phi = \rho F_\phi \\
Q_z =& F_z
\end{aligned}
$$

Where F_r, F_f and F_z are the components of the force along the increasing direction of r, f and z

Problem: 1.6- Masses m and m are connected by a light inextensible string which passes over a pulley of mass $2m$ and radius a. Write the Lagrangian and find the acceleration of the system.

Solution

The system has only one degree of freedom, and x see fig.(1.1) is taken as the generalized coordinate. The length of the string be l and the center of the pulley is taken as zero for potential energy

$$
\begin{aligned}
\text{K.E. of the system} \quad T =& \frac{1}{2}m\dot{x}^2 + m\dot{x}^2 + \frac{1}{2}I\omega^2 \\
=& \frac{3}{2}m\dot{x}^2 + \frac{1}{2}I\left(\frac{\dot{x}}{a}\right)^2 \\
\text{P.E. of the system} \quad V =& -mgx - 2mg(l-x) \\
\text{Lagrangian} \quad L =& \frac{3}{2}m\dot{x}^2 + \frac{I}{2a^2}\dot{x}^2 - mgx + 2mgl \\
\frac{\partial L}{\partial \dot{x}} =& \left(3m + \frac{I}{a^2}\right)\dot{x} \quad \frac{\partial L}{\partial x} = -mg
\end{aligned}
$$

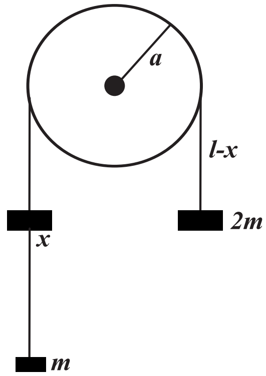

Fig. 1.1. A Pulley with a string carrying masses m and $2m$ at its end.

Substitution of these derivatives in Lagrange's equation gives the equation of motion:

$$\left(3m + \frac{I}{a^2}\right)\ddot{x} + mg = 0$$

$$\text{Acceleration} \quad \ddot{x} = -\frac{mg}{(3m + \frac{I}{a^2})} = -\frac{g}{4}$$

Since moment of inertia of the disc$= \frac{1}{2} \times 2ma^2 = ma^2$. Minus sign indicates mass m moves upward with the acceleration $g/4$.

Problem: 1.7- Prove that magnitude R of a position vector from an arbitrary origin for center of mass distribution $m_1, m_2, m_3, \cdots, m_n$ is given by,

$$M^2R^2 = \sum_i m_i^2 r_i^2 - \frac{1}{2}\sum_{ij} m_i m_j r_{ij}^2$$

Solution

Center of mass is defined as,

$$\vec{R} = \sum_i \frac{m_i \vec{r}_i}{M} \qquad \Rightarrow M\vec{R} = \sum_i m_i \vec{r}_i$$

$$\Rightarrow M\vec{R} \cdot M\vec{R} = \sum_i m_i \vec{r}_i \cdot \sum_j m_j \vec{r}_j$$

$$\Rightarrow \quad M^2R^2 = \sum_{i,j} m_i m_j \vec{r}_i \cdot \vec{r}_j \tag{1.1}$$

Now,

$$\begin{aligned}
\vec{r}_{ij} &= \vec{r}_i - \vec{r}_j \\
\Rightarrow \vec{r}_{ij} \cdot \rightarrow_{ij} &= (\vec{r}_i - \vec{r}_j) \cdot (\vec{r}_i - \vec{r}_j) \\
\Rightarrow \qquad \vec{r}^{\,2}_{ij} &= \vec{r}^{\,2}_i + \vec{r}^{\,2}_j - 2\vec{r}_i \cdot \vec{r}_j \\
\Rightarrow 2\vec{r}_i \cdot \vec{r}_j &= \vec{r}^{\,2}_i + \vec{r}^{\,2}_j - \vec{r}^{\,2}_{ij} \\
\Rightarrow \quad \vec{r}_i \cdot \vec{r}_j &= \frac{1}{2}(\vec{r}^{\,2}_i + \vec{r}^{\,2}_j) - \frac{1}{2}\vec{r}^{\,2}_{ij}
\end{aligned}$$

So equation (1.1) becomes,

$$\begin{aligned}
M^2R^2 &= \sum_{i,j} m_i m_j \left\{ \frac{1}{2}(r_i^2 + r_j^2) - \frac{1}{2}r_{ij}^2 \right\} = \sum_{i,j} m_i m_j \frac{1}{2}(r_i^2 + r_j^2) - \sum_{i,j} m_i m_j \frac{1}{2} r_{ij}^2 \\
\Rightarrow M^2R^2 &= \sum_{i} m_i m_j \frac{1}{2}(r_i^2 + r_j^2) - \sum_{i,j} m_i m_j \frac{1}{2} r_{ij}^2 = \sum_{i} m_i m_j \frac{1}{2}(2r_i^2) - \frac{1}{2}\sum_{i,j} m_i m_j r_{ij}^2 \\
\Rightarrow M^2R^2 &= \sum_{i} m_i^2 r_i^2 - \frac{1}{2}\sum_{i,j} m_i m_j \, r_{ij}^2
\end{aligned}$$

Hence Proved

Problem: 1.8- Prove that $grad\, S = \nabla S$

Solution

We know that,

$$\nabla S = \frac{\partial S}{\partial x}\hat{i} + \frac{\partial S}{\partial y}\hat{j} + \frac{\partial S}{\partial z}\hat{k}$$

Now,

$$\begin{aligned}
\nabla S \cdot \vec{dr} &= \left(\frac{\partial S}{\partial x}\hat{i} + \frac{\partial S}{\partial y}\hat{j} + \frac{\partial S}{\partial z}\hat{k} \right) \cdot (dx\hat{i} + dy\hat{j} + dz\hat{k}) \\
\Rightarrow \nabla S \cdot \vec{dr} &= \frac{\partial S}{\partial x}dx + \frac{\partial S}{\partial y}dy + \frac{\partial S}{\partial z}dz \qquad (1.2)
\end{aligned}$$

By definition,

$$\begin{aligned}
grad\, S &= \frac{\partial S}{\partial n}\hat{n} \qquad \Rightarrow grad\, S \cdot \vec{dr} = \frac{\partial S}{\partial n}\hat{n} \cdot \vec{dr} \\
\Rightarrow \quad grad\, S \cdot \vec{dr} &= \frac{\partial S}{\partial n} dr \cos\theta
\end{aligned}$$

Consider two surfaces very close together associated with constant values S_1 and S_2 of scalar field respectively.
Now,

$$\frac{dn}{dr} = \cos\theta \qquad \Rightarrow dn = dr\cos\theta$$
$$\Rightarrow \qquad grad\, S \cdot \vec{dr} = \frac{dS}{dn}dn = dS$$

In rectangular coordinates,

$$dS = \frac{\partial S}{\partial x}dx + \frac{\partial S}{\partial y}dy + \frac{\partial S}{\partial z}dz$$

Therefore,

$$grad\, S \cdot \vec{dr} = \frac{\partial S}{\partial x}dx + \frac{\partial S}{\partial y}dy + \frac{\partial S}{\partial z}dz \tag{1.3}$$

Comparing equations (1.2) and (1.3), we have

$$grad\, S \cdot \vec{dr} = \nabla S \cdot \vec{dr} \qquad \Rightarrow grad\, S = \nabla S$$

Problem: 1.9- (a)- Show that for a single particle with constant mass, the equation of motion implies

$$\frac{dT}{dt} = \vec{F} \cdot \vec{v}$$

Where T is kinetic energy, $\vec{F}$ is applied force vector and $\vec{v}$ is the velocit6y vector.
(b)- If the mass varies with time, the corresponding equation is,

$$\frac{d}{dt}(mT) = \vec{F} \cdot \vec{p}$$

Solution

(a)- Kinetic energy is,

$$T = \frac{1}{2}mv^2 = \frac{m^2v^2}{2m} \qquad \Rightarrow T = \frac{p^2}{2m} \qquad \Rightarrow mT = \frac{p^2}{2} \tag{1.4}$$

Now,

$$\frac{dT}{dt} = \frac{1}{2}m(2\vec{v})\cdot\left(\frac{d\vec{v}}{dt}\right) \qquad \Rightarrow \frac{dT}{dt} = m\vec{v}\cdot\vec{a}$$

$$\Rightarrow \quad \frac{dT}{dt} = m\vec{a}\cdot\vec{v} \qquad \Rightarrow \frac{dT}{dt} = \vec{F}\cdot\vec{v}$$

(b)- Differentiating equation (1.4) with respect to t,

$$\frac{d}{dt}(mT) = \vec{p}\cdot\frac{d\vec{p}}{dt} = \vec{p}\cdot\vec{F} \qquad \Rightarrow \frac{d}{dt}(mT) = \vec{F}\cdot\vec{p} \qquad \text{As required}$$

Problem: 1.10- Consider scattering of particles by a rigid sphere of radius R and calculate the differential and total cross-sections.

Solution Since the sphere is rigid, the potential outside the sphere is zero and that

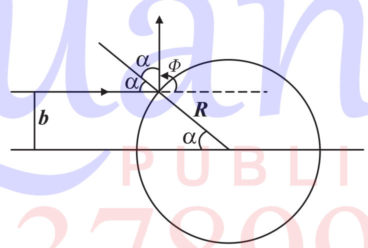

Fig. 1.2. Scattering by a rigid sphere

inside is. Fig.(1.2) illustrates the scattering by a rigid sphere. A particle with impact parameter $b > R$ will not be scattered.If $b < R$, due to the law of conservation of momentum and energy a particle incident at an angle a with the normal to the surface of the sphere will be scattered off on the other side of the normal at the same angle a (see Fig.(1.2))

we know,

$$\sigma(\phi) = -\frac{b\,db}{\sin\phi\,d\phi} \tag{1.5}$$

Now from figure,

$$\sin\alpha = \frac{b}{R} \quad and \quad \phi = \pi - 2\alpha$$
$$\alpha = \frac{\pi-\phi}{2} \quad or \quad \sin\alpha = \sin\frac{\pi-\phi}{2} = \cos\frac{\phi}{2}$$

Equating the two expressions for $\sin\alpha$

$$b = R\cos\frac{\phi}{2}$$

Substituting this value of b in Eq.(3.6)

$$\sigma(\phi) = -\frac{b}{\sin\phi}\frac{db}{d\phi} = \frac{R^2}{4}$$

Which is independent of f and incident energy.

$$\sigma_T = \int_{4\pi}\sigma(\Omega)d\Omega = 2\pi\int_0^\pi \sigma(\phi)\sin\phi\, d\phi$$
$$= 2\pi\frac{R^2}{4}[-\cos\phi]_0^\pi = \pi R^2$$

Problem: 1.11- A projectile is launched with muzzle velocity of 1800 $miles/h$ at an angle of $60'$ with horizontal and lands on same plane. Find,

(a)- Max height reached.

(b)- Time to reach maximum height.

(c)- Total time of flight.

(d)- Range of projectile.

Solution

$$\text{Muzzle velocity} = v_\circ = 1800\, miles/h = \frac{1800\times1760\times3}{60\times60}\, ft/s = 2640\, ft/s$$
$$\text{Angle of projection} = \theta = 60^\circ$$

(a)- Max height reached is,

$$H = \frac{v_\circ^2 \sin^2\theta}{2g} = \frac{(2460\,ft/s)^2 \times (\sin 60^\circ)^2}{2 \times 32\,ft/s^2} = 81675\,ft$$

(b)- Time to reach maximum height is,

$$t_m = \frac{v_\circ \sin\theta}{g} = \frac{2460\,ft/s \times \sin 60^\circ}{32\,ft/s^2} = 71.5\,s$$

(c)- Total time of flight is,

$$t_f = 2t_m = 2 \times 71.5\,s = 143\,s$$

(d)- Range of projectile is,

$$R = \frac{v_\circ^2 \sin 2\theta}{g} = \frac{(2460\,ft/s)^2 \times \sin 120^\circ}{32\,ft/s^2} = 188614.800\,ft$$
$$= \frac{188614.800}{1760 \times 3}\,miles = 35.72\,miles$$

Problem: 1.12- Masses of 1, 2 and 3 kg are located at positions $4\hat{j} + 3\hat{k}$ and $2\hat{i} + 2\hat{k}$ respectively. If their velocities are $7\hat{i}$, $-6\hat{j}$ and $-3\hat{k}$, find the position and velocity of the center of mass. Also, find the angular momentum of the system with respect to the origin.

Solution

Radius vector of the center of mass

$$R = \sum_i \frac{m_i r_i}{M} = \frac{1(\hat{i} + \hat{j} + \hat{k}) + 2(4\hat{j} + 3\hat{k}) + 3(2\hat{i} + 2\hat{k})}{6}$$
$$= \frac{(7\hat{i} + 9\hat{j} + 13\hat{k})}{6}$$

Velocity of the center of mass

$$v = \frac{\sum_i m_i v_i}{M} = \frac{1 \times 7\hat{i} + 2(-6\hat{j}) + 3(-3\hat{i})}{6}$$
$$= \frac{-2\hat{i} - 12\hat{j}}{6} = \frac{-\hat{i} - 6\hat{j}}{3}$$

The angular momentum vector about the origin

$$\begin{aligned}L =&\sum_i r_i \times m_i v_i\\ =&(\hat{i}+\hat{j}+\hat{k})\times 7\hat{i} + (4\hat{j}+3\hat{k})\times 2(-6\hat{j}) + (2\hat{i}+2\hat{k})\times 3(-3\hat{i})\\ =&7\hat{j}-7\hat{k}+36\hat{i}-18\hat{j} = 36\hat{i}-11\hat{j}-7\hat{k}\end{aligned}$$

Problem: 1.13- Particles of masses 1, 2 and 4 kg move under a force such that their position vectors at time t are respectively $r_1 = 2\hat{i} + 4t^2\hat{k}$, $r_2 = 4t\hat{i} - \hat{k}$, and $r_3 = (\cos\pi t)\hat{i} + (\sin\pi t)\hat{j}$. Find the angular momentum of the system about the origin at $t = 1$ s.

Solution

The angular momentum L is given by

$$\begin{aligned}L =&\sum_i r_i \times m_i \dot{r}_i\\ =&(2\hat{i}+4t^2\hat{k})\times 8t\hat{k} + (4t\hat{i}-\hat{k})\times 8\hat{i} + [(\cos\pi t)\hat{i} + (\sin\pi t)\hat{j}]\times 4\pi[(-\sin\pi t)\hat{j} + \cos\pi t)\hat{i}]\\ =&-16t\hat{j} - 8\hat{j} + 4\pi(\cos^2\pi t + \sin^2\pi t)\hat{k}\\ (L)_{t=1s} =&-24\hat{j} + 4\pi\hat{k}\end{aligned}$$

Problem: 1.14- Consider a system of N particles with masses $m_1, m_2, m_3 \cdots m_N$ located at cartesian coordinates $r_1, r_2, \cdots r_N$ acted upon by forces derivable from a potential function $v(r_1, r_2, \cdots, r_N)$. Show that Lagrange equations of motion reduce directly to Newton's second law.

Solution

$$\text{The kinetic energy} T = \sum_{i=1}^{N}\frac{1}{2}m_i\dot{r}_i^2$$

$$\text{Lagrangian} \quad L = T - V = \frac{1}{2}\sum_i m_i\dot{r}_i^2 - V(r_1, r_2, \cdots, r_N)$$

$$\frac{\partial L}{\partial r_i} = -\frac{\partial V}{\partial r_i} \qquad \frac{\partial L}{\partial \dot{r}_i} = m\dot{r}_i \qquad F_i = -\frac{\partial V}{\partial r_i}$$

Identifying the rectangular co-ordinates as the generalized co-ordinates, Lagrange's equation can be written as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}_i}\right) - \frac{\partial L}{\partial r_i} = 0 \qquad i = 1, 2, \cdots, N$$

Substituting the above values

$$\frac{d}{dt}(m_i \dot{r}_i) + \frac{\partial L}{\partial r_i} = 0 \qquad i = 1, 2, \cdots, N$$
$$m_i \ddot{r}_i = -\frac{\partial L}{\partial r_i} = F_i \qquad i = 1, 2, \cdots, N$$

Which is familiar form of Newton's second law.

Problem: 1.15- A disc rolling on a horizontal xy-plane is constrained to move such that the plane of disc is always vertical. Show that the constraint in this example is non-holonomic

Solution

Consider a disk is rolling on horizontal xy-plane constrained to move such that plane of disc is always vertical. Let a be radius of disk and let ϕ be angular displacement made by disk and θ be angle which the axis of disk makes with x-axis.For angular displacement,

$$s = a\phi \qquad \Rightarrow \dot{s} = a\dot{\phi} \qquad \Rightarrow v = a\dot{\phi}$$

The components of velocity are;

$$v_x = v\cos\left(\frac{\pi}{2} - \theta\right), \qquad v_y = -v\sin\left(\frac{\pi}{2} - \theta\right)$$
$$\Rightarrow \qquad v_x = v\sin\theta, \qquad v_y = -v\cos\theta$$

Negative sign in v_y is due to fact that y-component is along negative y-axis. So we can write,

$$\dot{x} = v\sin\theta, \qquad \dot{y} = -v\cos\theta$$
$$\Rightarrow \qquad \dot{x} = a\sin\theta\dot{\phi}, \qquad \dot{y} = -a\cos\theta\dot{\phi}$$
$$\Rightarrow \qquad \frac{dx}{dt} = a\sin\theta\frac{d\phi}{dt}, \qquad \frac{dy}{dt} = -a\cos\theta\frac{d\phi}{dt}$$
$$\Rightarrow \qquad dx - a\sin\theta d\phi = 0 \qquad \& \qquad dy + a\cos\theta d\phi = 0$$

Neither of above equation can be integrated, so the constraints are non-holonomic.

Chapter 2

Variational Principles

SOLVED PROBLEMS

Problem: 2.1- Given a mass spring system consisting of a mass and linear spring of stiffness k as shown in the Fig.(2.1). Find the equation of motion using Hamiltonian procedure. Assume that the displacement x is measured from unstressed position of string.

Solution

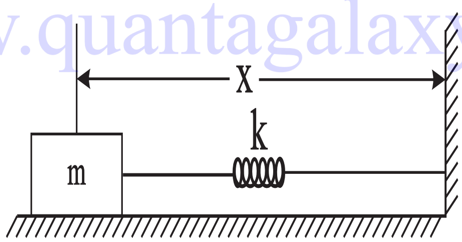

Fig. 2.1. The schematic picture which shows the mass spring system.

Let us find K.E. and P.E., so

$$T = \frac{1}{2}m\dot{x}^2$$

$$\text{and} \qquad V = \frac{1}{2}kx^2$$

Now, the Lagrangian is defined as:

$$
\begin{aligned}
L &= T - V \\
L &= \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2
\end{aligned}
$$

Also, Hamilton's Principle is defined as:

$$
\begin{aligned}
\delta \int_{t_1}^{t_2} L dt &= 0 \\
\text{or} \qquad \int_{t_1}^{t_2} \delta L dt &= 0 \\
\int_{t_1}^{t_2} \delta \left(\frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 \right) dt &= 0 \\
\int_{t_1}^{t_2} \left(\frac{1}{2}m\delta\dot{x}^2 - \frac{1}{2}k\delta x^2 \right) dt &= 0 \\
\int_{t_1}^{t_2} \left(\frac{1}{2}m(2\dot{x})\delta\dot{x} - \frac{1}{2}k(2x)\delta x \right) dt &= 0 \\
\int_{t_1}^{t_2} (m\dot{x}\delta\dot{x} - kx\delta x)\, dt &= 0 \\
\text{or} \qquad \int_{t_1}^{t_2} m\dot{x}\frac{d}{dt}(\delta x) dt - \int_{t_1}^{t_2} kx\delta x dt &= 0
\end{aligned}
$$

Evaluating 1$^{\text{st}}$ integrate by parts, we have

$$
\begin{aligned}
\int_{t_1}^{t_2} m\dot{x}\frac{d}{dt}(\delta x) dt &= m\dot{x}\delta x \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \delta x m\ddot{x} dt \\
\int_{t_1}^{t_2} m\dot{x}\frac{d}{dt}(\delta x) dt &= m\dot{x}\left[\delta x(t_2) - \delta x(t_1)\right] - \int_{t_1}^{t_2} \delta x m\ddot{x} dt
\end{aligned}
$$

$$\int_{t_1}^{t_2} m\dot{x}\frac{d}{dt}(\delta x)dt = m\dot{x}\left[\delta x(t_1) - \delta x(t_1)\right] - \int_{t_1}^{t_2} \delta x m\ddot{x}dt \qquad \because \delta x(t_1) = 0 = \delta x(t_2)$$

$$\int_{t_1}^{t_2} m\dot{x}\frac{d}{dt}(\delta x)dt = m\dot{x}\left[0\right] - \int_{t_1}^{t_2} \delta x m\ddot{x}dt$$

$$\int_{t_1}^{t_2} m\dot{x}\frac{d}{dt}(\delta x)dt = -\int_{t_1}^{t_2} \delta x m\ddot{x}dt$$

Therefore, we have

$$-\int_{t_1}^{t_2} \delta x m\ddot{x}dt - \int_{t_1}^{t_2} kx\delta xdt = 0$$

$$\text{or} \quad \int_{t_1}^{t_2} \delta x m\ddot{x}dt + \int_{t_1}^{t_2} kx\delta xdt = 0$$

$$\int_{t_1}^{t_2} (m\ddot{x} + kx)\,\delta xdt = 0$$

If an integral is zero, its integrand can also be zero. Therefore,

$$(m\ddot{x} + kx)\,\delta x = 0$$

$$\text{But,} \quad \delta x \neq 0$$

$$\text{So,} \quad m\ddot{x} + kx = 0$$

$$\text{or,} \quad ma + kx = 0 \qquad \because F = ma = m\ddot{x}$$

$$ma = -kx$$

Which is the equation of motion. This equation can also be obtained by using Newton's law of motion or Lagrange's equation.

Problem: 2.2- Obtain Hamilton's equation for a simple pendulum. Hence obtain an expression for its period.

Solution

In simple pendulum we use q as the generalized coordinate. For evaluating potential energy, the energy corresponding to the mean position is taken as zero. The velocity of the bob $v = l\dot{\theta}$.

$$\text{Kinetic energy} \quad T = \frac{1}{2} m l^2 \dot{\theta}^2$$

$$\text{Potential energy} \quad V = mgl(1 - \cos\theta)$$

$$L = T - V = \frac{1}{2} m l^2 \dot{\theta}^2 - mgl(1 - \cos\theta) \tag{2.1}$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m l^2 \dot{\theta} \quad or \quad \dot{\theta} = \frac{p_\theta}{m l^2}$$

$$\text{Hamiltonian} \quad H(\theta, p_\theta) = \dot{\theta} p_\theta - L$$

$$= \frac{1}{2 m l^2} p_\theta^2 + mgl(1 - \cos\theta) \tag{2.2}$$

Hamilton's equations are;

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{m l^2} \quad \dot{p}_\theta = -\frac{\partial H}{\partial \theta} = -mgl\sin\theta \tag{2.3}$$

$$\ddot{\theta} = \frac{\dot{p}_\theta}{m l^2} = -\frac{g\sin\theta}{l}$$

Since θ is small, $\sin\theta \cong \theta$ and above equation reduces to

$$\ddot{\theta} = \frac{-g\theta}{l} \tag{2.4}$$

The motion is simple harmonic, and the period T is given by

$$T = 2\pi\sqrt{\frac{l}{g}} \tag{2.5}$$

Problem: 2.3- A mass m is suspended by a massless spring of spring constant k. The suspension point is pulled upwards with constant acceleration a_0. Find the Hamiltonian of the system, Hamilton's equations of motion and the equation of motion.

Solution

Let the vertical be the z-axis. As the acceleration due to gravity is downwards, taking the net acceleration as $(g - a_0)$.

$$\text{Potential energy} \qquad V = \frac{1}{2}kz^2 + m(g - a_0)z$$

$$\text{Kinetic energy} \qquad T = \frac{1}{2}m\dot{z}^2$$

$$L = \frac{1}{2}m\dot{z}^2 - \frac{1}{2}kz^2 - m(g - a_0)z \tag{2.6}$$

$$p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z} \quad or \quad \dot{z} = \frac{p_z}{m}$$

$$H = p_z\,\dot{z} - L = \frac{p_z^2}{2m} + \frac{1}{2}kz^2 + m(g - a_0)z \tag{2.7}$$

Hamilton's equation are

$$\dot{z} = \frac{\partial H}{\partial p_z} = \frac{p_z}{m} \tag{2.8}$$

$$\dot{p}_z = -\frac{\partial H}{\partial z} = -kz - m(g - a_0) \tag{2.9}$$

The equation of motion is

$$\ddot{z} = \frac{1}{m}\dot{p}z = \frac{1}{m}[-kz - m(g - a_0)]$$

$$m\ddot{z} = -kz - m(g - a_0) \tag{2.10}$$

Problem: 2.4- A particle of mass m moves in three dimensions under the action of a central conservative force with potential energy $V(r)$. Then,

(i)- Find the Hamiltonian function in spherical polar coordinates.

(ii)- Show that f is an ignorable coordinate.

(iii)- Obtain Hamilton's equation of motion.

(iv)- Express the quantity $p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}$ or $\dot{r} = \frac{p_r}{m}$ in term of generalized momenta.

Solution

(i):

$$
\begin{aligned}
\text{Kinetic energy} \quad T =& \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2) \\
L =& \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2) - V(r) \\
p_r =& \frac{\partial L}{\partial \dot{r}} = m\dot{r} \quad or \quad \dot{r} = \frac{p_r}{m} \\
p_\theta =& \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \quad or \quad \dot{\theta} = \frac{p_\theta}{mr^2} \\
p_\phi =& \frac{\partial L}{\partial \dot{\phi}} = mr^2\sin^2\theta\dot{\phi} \quad or \quad \dot{\phi} = \frac{p_\phi}{mr^2\sin^2\theta} \\
H =& \sum_i p_i\dot{q}_i - L = p_r\dot{r} + p_\theta\dot{\theta} + p_\phi\dot{\phi} - L
\end{aligned}
$$

Substituting the values of $\dot{r}, \dot{\theta}$ and $\dot{\phi}$, we have,

$$
H = \frac{1}{2m}\left[p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2\sin^2\theta}\right] + V(r)
$$

(ii): The coordinates f is not appearing in the Hamiltonian. Hence, it is an ignorable coordinate.

(iii): Hamilton's canonical equations will be six in number as there are three generalized coordinates. They are,

$$
\begin{aligned}
\dot{p}_r &= -\frac{\partial H}{\partial r} = \frac{1}{mr^3}\left(p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta}\right) - \frac{dV(r)}{dr} & \dot{r} &= \frac{\partial H}{\partial p_r} = \frac{p_r}{m} \\
\dot{p}_\theta &= -\frac{\partial H}{\partial \theta} = \frac{1}{mr^2}\frac{p_\phi^2\cos\theta}{\sin^3\theta} & \dot{\theta} &= \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2} \\
\dot{p}_\phi &= -\frac{\partial H}{\partial \phi} = 0 & \dot{\phi} &= \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{mr^2\sin^2\theta}
\end{aligned}
$$

(iv):

$$
\begin{aligned}
l^2 =& m^2r^4\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right) = m^2r^4\left(\frac{p_\theta^2}{m^2r^4} + \frac{\sin^2\theta p_\phi^2}{m^2r^4\sin^4\theta}\right) \\
=& p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta}
\end{aligned}
$$

Problem: 2.5- Find Lagrange of equation of motion of a simple harmonic oscillator on which a non-conservative force $F_\circ \sin \omega t$ is applied.

Solution

consider a mass m attached to a spring of spring constant k. Suppose at any time t it is at a distance x from fixed point O. Since system can be completely specified by one coordinate x so there is only one Lagrange equation. Kinetic and potential energies are

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\dot{x}^2 \quad \& \quad V = \frac{1}{2}kx^2$$

Lagrangian is,

$$L = T - V = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$

For coordinate x, Lagrange equation is,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = Q \qquad \Rightarrow \frac{d}{dx}(m\dot{x}) + kx = F_\circ \sin \omega t$$

$$\Rightarrow \quad m\ddot{x} + kx = F_\circ \sin \omega t \qquad \Rightarrow \ddot{x} + \frac{k}{m}x = \frac{F_\circ}{m} \sin \omega t$$

Problem: 2.6- Lagrangian for motion of a particle in electromagnetic field is

$$L = \frac{1}{2}m\dot{x}^2 + Q(\dot{x} \cdot A_\phi)$$

Where Q is the particle's charge, $A(x,t)$ is the magnetic vector potential and $\phi(x,t)$ is the electrostatic potential. Find Lagrange equation of motion.

Solution

Here is only one generalized coordinate x, so there is only one equation of motion, Action is,

$$S = \int L dt = \int \left[\frac{1}{2}m\dot{x}^2 + Q(\dot{x} \cdot A - \phi)\right] dt \qquad \longrightarrow (a)$$

Lagrange equation is,

$$\frac{d}{dt}(m\dot{x} + QA) + Q\nabla(\phi - \dot{x} \cdot A) = 0 \longrightarrow (b)$$

Here derivative with respect to t is along the path, so

$$\frac{dA}{dt} = \frac{\partial A}{\partial t} + (\dot{x} \cdot \nabla)A \longrightarrow (c)$$

Electric field

$$E = -\nabla\phi - \frac{\partial A}{\partial t}$$

So equation (b) becomes,

$$m\ddot{x} = Q[E + \nabla(\dot{x} \cdot A) - (\dot{x} \cdot \nabla)A]$$

Now

$$\dot{x} \times B = \dot{x} \times (\nabla \times A) = \nabla(\dot{x} \cdot A) - (\dot{x} \cdot \nabla)A$$

The above equation simplifies to,

$$m\ddot{s} = Q(E + \dot{x} \times B)$$

Chapter 3

Two Body Central Force Problems

SOLVED PROBLEMS

Problem: 3.1- A particle moves in a circular orbit of diameter b in a central force field. If the center of attraction is on the circumference itself, find the law of force.

Solution

In a central force field, the differential equation of the orbit, is given by,

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)+\frac{1}{r}=-\frac{m}{L^2}r^2F(r) \tag{3.1}$$

Here, O is the center of force, and A is the position of the particle. the co-ordinates of the particle are r and q. From the figure

$$r = b\cos\theta \tag{3.2}$$

$$\frac{d}{d\theta}\left(\frac{1}{r}\right) = \frac{d}{d\theta}\left(\frac{\sec\theta}{b}\right) = \frac{1}{b}\sec\theta\tan\theta$$

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) = \frac{1}{b}(\sec\theta\tan^2\theta+\sec^3\theta) \tag{3.3}$$

Substituting Eq.(3.3) in Eq.(3.1), we get

$$\frac{1}{b}(\sec\theta\tan^2\theta+\sec^3\theta)+\frac{\sec\theta}{b}$$

$$= -\frac{m}{L^2}b^2\cos^2\theta F(r) \tag{3.4}$$

$$\frac{1}{b}[\sec\theta(\sec^2\theta - 1) + \sec^3\theta] + \frac{\sec\theta}{b} = -\frac{m}{L^2}b^2\cos^2\theta F(r)$$

$$\frac{2\sec^3\theta}{b} = -\frac{m}{L^2}b^2\cos^2\theta F(r)$$

$$F(r) = \frac{-2L^2\sec^5\theta}{mb^3} = \frac{-2L^2b^2}{mr^5} = \frac{K}{r^5} \tag{3.5}$$

Where K is a constant.

Problem: 3.2- A spacecraft in a circular orbit of radius r_c around the earth was put in an elliptical orbit by firing a rocket. If the speed of the spacecraft increased by 12.5% by the sudden firing of the rocket,

(i) What is the equation of the new orbit?

(ii) What is its eccentricity?

(iii) What is the apogee distance?

Solution

Let v_c be the speed in the circular orbit. The speed after firing of rocket

$$v_\circ = v_c + 0.125v_c = 1.125v_c$$

(i)- the equation of orbit is given by,

$$r = \frac{(1.125)^2 r_c}{1 + [(1.125)^2 - 1]\cos\theta} = \frac{1.27r_c}{1 + 0.27\cos\theta}$$

(ii)- Eccentricity

$$\epsilon = \left(\frac{v_p}{v_\circ}\right)^2 - 1 = (1.125)^2 - 1 = 0.27$$

(iii)- At the apogee, $\theta = \pi$ and r is $r_{\max}$

$$r_{\max} = \frac{1.27 r_c}{1 - 0.27} = 1.74 r_c$$

Problem: 3.3- Consider scattering of particles by a rigid sphere of radius R and calculate the differential and total cross-sections.

Solution

Since the sphere is rigid, the potential outside the sphere is zero and that inside the scattering by a rigid sphere. A particle with impact parameter $b > R$ will not be scattered.If $b < R$, due to the law of conservation of momentum and energy a particle incident at an angle a with the normal to the surface of the sphere will be scattered off on the other side of the normal at the same angle a (see Fig.(1.2))

we know,

$$\sigma(\phi) = -\frac{b\,db}{\sin\phi\,d\phi} \tag{3.6}$$

Now from figure,

$$\sin\alpha = \frac{b}{R} \quad and \quad \phi = \pi - 2\alpha$$

$$\alpha = \frac{\pi - \phi}{2} \quad or \quad \sin\alpha = \sin\frac{\pi - \phi}{2} = \cos\frac{\phi}{2}$$

Equating the two expressions for $\sin\alpha$

$$b = R\cos\frac{\phi}{2}$$

Substituting this value of b in Eq.(3.6)

$$\sigma(\phi) = -\frac{b}{\sin\phi}\frac{db}{d\phi} = \frac{R^2}{4}$$

Which is independent of f and incident energy.

$$\sigma_T = \int_{4\pi} \sigma(\Omega)d\Omega = 2\pi\int_0^{\pi} \sigma(\phi)\sin\phi\,d\phi$$

$$=2\pi\frac{R^2}{4}[-\cos\phi]_0^{\pi} = \pi R^2$$

Problem: 3.4- Find the law of force if a particle under central force moves along the curve $r = a(1+\cos\theta)$.

Solution

The differential equation of the orbit is

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)+\frac{1}{r} \tag{3.7}$$

$$= -\frac{mr^2}{L^2}F(r) \tag{3.8}$$

$$\frac{d}{d\theta}\left(\frac{1}{r}\right) = \frac{d}{d\theta}\left(\frac{1}{a(1+\cos\theta)}\right) \tag{3.9}$$

$$\frac{d}{d\theta}\left(\frac{1}{r}\right) = \frac{\sin\theta}{a(1+\cos\theta)^2} \tag{3.10}$$

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) = \frac{d}{d\theta}\left\{\frac{\sin\theta}{a(1+\cos\theta)^2}\right\} \tag{3.11}$$

$$= \frac{\cos\theta}{a(1+\cos\theta)^2}+\frac{2\sin^2\theta}{a(1+\cos\theta)^3}$$

$$= \frac{a\cos\theta}{a^2(1+\cos\theta)^2}+\frac{2a^2(1-\cos^2\theta)}{a^3(1+\cos\theta)^3} \tag{3.12}$$

$$= \frac{r-a}{r^2}+\frac{2a^2-2(r-a)^2}{r^3}$$

$$= \frac{r-a}{r^2}+\frac{-2r^2+4ar}{r^3} \tag{3.13}$$

$$= \frac{1}{r}-\frac{a}{r^2}-\frac{2}{r}+\frac{4a}{r^2}$$

$$= -\frac{1}{r}+\frac{3a}{r^2} \tag{3.14}$$

Substituting Eq.(3.14) in Eq.(3.8)

$$-\frac{1}{r}+\frac{3a}{r^2}+\frac{1}{r} = -\frac{mr^2}{L^2}F(r)$$

$$F(r) = -\frac{3aL^2}{mr^4}$$

Which is the law of force.

Problem: 3.5- For circular and parabolic orbits in an attractive $\frac{1}{r}$ potential having the same angular momentum, prove that the speed of the particle at any time in the parabolic orbit is $\sqrt{2}$ times the speed in circular orbit passing through the same point.

Solution

We know that the solution to equation of law of force is,

$$\frac{1}{r} = \frac{mk}{l^2}[1 + \epsilon \cos(\theta - \theta')]$$

The speed of a particle in a circular orbit is,

$$v_c^2 = r^2\dot{\theta}^2 = r^2\left(\frac{l^2}{m^2r^4}\right) \qquad \Rightarrow v_c = \frac{1}{mr}$$

In term of k, its equal to

$$v_c = \frac{l}{mr} = \frac{\sqrt{mrk}}{mr} = \sqrt{\frac{k}{mr}}$$

The speed of a particle in a parabolic path,

$$v_p^2 = \dot{r}^2 + r^2\dot{\theta}^2$$
$$= \left\{\frac{d}{dt}\left(\frac{l^2}{mk(1+\cos\theta)}\right)\right\}^2 + r^2\dot{\theta}^2$$
$$v_p^2 = \dot{r}^2 + r^2\dot{\theta}^2 = \frac{l^2\dot{\theta}}{mk(1+\cos\theta)^2}\sin\theta + r^2\dot{\theta}^2$$
$$= r^2\dot{\theta}^2\left(\frac{\sin^2\theta}{(1+\cos\theta)^2} + 1\right)$$
$$v_p^2 = r^2\dot{\theta}^2\left(\frac{2+2\cos\theta}{(1+\cos\theta)^2}\right) = \frac{2r^2\dot{\theta}^2}{1+\cos\theta}$$

Using $k = \frac{l^2}{mr}$, we have

$$r = \frac{l^2}{mk(1+\cos\theta)} \qquad \& \qquad \dot{\theta}^2 = \frac{l^2}{m^2r^4}$$

We have

$$v_p^2 = \frac{2l^2r^2mkr}{m^2r^4l^2} = \frac{2k}{mr}$$

For the speed of parabola, we have

$$v_p = \sqrt{2}\sqrt{\frac{k}{mr}}$$

Thus,

$$v_p = \sqrt{2}\, v_c$$

Chapter 4

Kinematics of Rigid Body

SOLVED PROBLEMS

Problem: 4.1- A body moves about a point O under no force. The principal moments of inertia at O being 3A, 5A and 6A. Initially the angular velocity has components $w_1 = w, \;\; w_2 = 0$ and $w_3 = 2$ about the corresponding principal axes. Show that at time t,

$$\omega_2 = \frac{3\omega}{\sqrt{5}} \tan \frac{\omega t}{\sqrt{5}}$$

$$if \int \frac{dx}{p^2 - x^2} = \frac{1}{p} \tanh^{-1} \left(\frac{x}{p} \right)$$

Solution

In the torque-free case, the Euler's equations are

$$I_1 \dot{\omega}_1 = \omega_2 \omega_3 (I_2 - I_3) \tag{4.1}$$

$$I_2 \dot{\omega}_2 = \omega_1 \omega_3 (I_3 - I_1) \tag{4.2}$$

$$I_3 \dot{\omega}_3 = \omega_1 \omega_2 (I_1 - I_2) \tag{4.3}$$

Replacing the principal moments of inertia I_1, I_2, I_3 by $3A, 5A$ and $6A$, respectively

$$3\dot{\omega}_1 = -\,\omega_2\omega_3 \tag{4.4}$$

$$5\dot{\omega}_2 = 3\omega_3\omega_1 \tag{4.5}$$

$$6\dot{\omega}_3 = -\,2\omega_1\omega_2 \tag{4.6}$$

Multiplying Eq.(4.6) by $3\omega_1$ and Eq.(4.5) by ω_2 and adding.

$$9\omega_1\dot{\omega}_1 + 5\omega_2\dot{\omega}_2 = 0$$

Integrating and applying the initial conditions

$$9\omega_1^2 + 5\omega_2^2 = Constant$$

$$9\omega_1^2 + 5\omega_2^2 = 9\omega^2 \tag{4.7}$$

Similarly from Eqs.(4.4) and (4.6)

$$\omega_1^2 = \omega_3^2 \tag{4.8}$$

Using Eqs.(4.8), (4.5) and (4.7), we have

$$5\dot{\omega}_2 = 3\omega_1^2 = 3\omega^2 - \frac{5\omega_2^2}{3} \quad or \quad \dot{\omega}_2 = \frac{9\omega^2 - 5\omega_2^2}{15}$$

Integrating

$$t = 15\int \frac{d\omega_2}{9\omega^2 - 5\omega_2^2} = 3\int \frac{d\omega_2}{\left(\frac{9}{5}\right)\omega^2 - \omega_2^2}$$

$$= \frac{\sqrt{5}}{\omega}\tanh^{-1}\left(\frac{\sqrt{5}\omega_2}{3\omega}\right)$$

$$\omega_2 = \frac{3\omega}{\sqrt{5}}\tanh\left(\frac{\omega t}{\sqrt{5}}\right)$$

Problem: 4.2- In the absence of external torque on a body, prove that

(i)- The kinetic energy is constant.

(ii)- The magnitude of the square of the angular momentum (L^2) is constant.

Solution

According to Simpler form of Euler's equations, which are,

$$I_1\dot{\omega}_1 = \omega_2\omega_3(I_2 - I_3) \tag{4.9}$$

$$I_2\dot{\omega}_2 = \omega_3\omega_1(I_3 - I_1) \tag{4.10}$$

$$I_3\dot{\omega}_3 = \omega_1\omega_2(I_1 - I_2) \tag{4.11}$$

(i)- Multiplying the equation (4.9) by w_1, (4.10) by w_2 and the equation (4.11) by w_3, and adding, we get

$$\begin{aligned} I_1\omega_1\dot{\omega}_1 + I_2\omega_2\dot{\omega}_2 + I_3\omega_3\dot{\omega}_3 =& 0 \\ \frac{1}{2}\frac{d}{dt}[I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2] =& 0 \end{aligned}$$

The quantity inside the square bracket is kinetic energy $2T$, that is

$$\frac{d}{dt}(T) = 0 \qquad or \qquad T \text{ is a constant}$$

(ii)-

$$\begin{aligned} L^2 =& (I_1\omega_1 + I_2\omega_2 + I_3\omega_3) \\ & \cdot (I_1\omega_1 + I_2\omega_2 + I_3\omega_3) \\ L^2 =& I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2 \end{aligned}$$

Multiplying the equation (4.9) by $I_1\omega_1$, equation (4.10) by $I_2\omega_2$ and the equation (4.11) by $I_3\omega_3$ and adding, we get

$$\begin{aligned} I_1^2\omega_1\dot{\omega}_1 + \text{i}_2^2\omega_2\dot{\omega}_2 + \text{i}_3^2\omega_3\dot{\omega}_3 =& 0 \\ \frac{1}{2}\frac{d}{dt}[I_1^2\omega_1^2 + I_2^2\omega_2^2 + {I_3}^2\omega_3^2] =& 0 \\ \frac{d}{dt}L^2 =& 0 \\ L^2 =& \text{Constant} \end{aligned}$$

Problem: 4.3- If w_3 is the angular velocity of a freely rotating symmetric top about its symmetry axis, show that the symmetry axis rotates about the space-fixed z-axis with angular frequency $\dot{\phi} = \frac{(2I_1 - I_3)\omega_3}{I_1 \cos\theta}$, where q and f are Euler's angles.

Solution

According to Euler's geometrical equations, which are,

$$\omega_1 = \omega_x = \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi \tag{4.12}$$

$$\omega_2 = \omega_y = \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi \tag{4.13}$$

$$\omega_3 = \omega_z = \dot{\phi}\cos\theta + \dot{\psi} \tag{4.14}$$

From the equation Eq.(4.14), we have

$$\omega_3 = \dot{\phi}\cos\theta + \dot{\psi}$$

In the force-free motion of a symmetric top we have seen that the angular velocity vector w of the top precesses in a cone about the body symmetry axis with an angular frequency k given by

$$k = \frac{(I_3 - I_1)\omega_3}{I_1}$$

This angular frequency is the same as $\dot{\psi}$ which is also directed along the symmetry axis. Substituting this value of $\dot{\psi}$ in the expression for w_3 and simplifying, we get

$$\dot{\phi} = \frac{(2I_1 - I_3)\omega_3}{I_1\cos\theta}$$

Problem: 4.4- Consider a thin rod of length l and mass m pivoted about one end. calculate the moment of inertia, Find the point at which, if all the masses were concentrated, the moment of inertia about that pivot axis would be the same as the real moment of inertia. The distance from this point to the pivot is called the radius of gyration.

Solution

The linear density of the rod is

$$\rho_l = \frac{m}{l}$$

For the origin at one end of the rod, the moment of inertia is,

$$I = \int_0^l \rho_l x^2 dx = \frac{m}{l}\frac{l^3}{3} = \frac{m}{3}l^2 \longrightarrow (a)$$

If all of the masses were concentrated at the point which is at distance α from the origin, the moment of inertia would be

$$I = ma^2 \longrightarrow (b)$$

Equating equations (a) and (b), we find

$$\alpha = \frac{l}{\sqrt{3}}$$

This is the radius of gyration.

Problem: 4.5- Solve the Hamilton-Jacobi equation for the system whose Hamiltonian is given by

$$H = \frac{p^2}{2} - \frac{\mu}{q}$$

Solution

The Hamilton-Jacobi equation is,

$$\frac{\partial S}{\partial t} + \frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 + mgx = 0$$

We assume

$$S = f(t) + \phi(q)$$

Now above equation gives,

$$\frac{\partial f}{\partial t} + \frac{1}{2}\left(\frac{\partial \phi}{\partial q}\right)^2 - \frac{\mu}{q} = 0$$

This equation can be satisfied by writing,

$$\frac{\partial f}{\partial t} = \frac{\mu}{q} - \frac{1}{2}\left(\frac{\partial \phi}{\partial q}\right)^2 = \frac{\mu}{\alpha}$$

Where α is a constant.

$$\begin{aligned} f(t) =& \frac{\mu}{\alpha}t, \\ \phi(q) =& \sqrt{2\mu\alpha}\arcsin\sqrt{\frac{q}{\alpha}} + \left(\frac{2\mu q(\alpha - q)}{\alpha}\right)^{\frac{1}{2}} \end{aligned}$$

Chapter 5

The Rigid Body Equations of Motion

SOLVED PROBLEMS

Problem: 5.1- A body can rotate freely about the principal axis corresponding to the principal moment of inertia I_3. If it is given a small displacement, show that the rotation will be oscillatory if I_3 is either the largest or the smallest of the three principal moments of inertia.

Solution

As we have Simpler form of Euler's equations,

$$I_1\dot{\omega}_1 = \omega_2\omega_3(I_2 - I_3) \tag{5.1}$$

$$I_2\dot{\omega}_2 = \omega_3\omega_1(I_3 - I_1) \tag{5.2}$$

$$I_3\dot{\omega}_3 = \omega_1\omega_2(I_1 - I_2) \tag{5.3}$$

Since the displacement is small, we may take w_1 and w_2 as small and the product w_1w_2 may be neglected. From the equation (5.3) we get,

$$\dot{\omega}_3 = 0 \qquad or \qquad \omega_3 = Constant$$

From the equation (5.1), we have

$$\ddot{\omega}_1 = \frac{\omega_3(I_2 - I_3)}{I_1}\dot{\omega}_2$$

Substituting the value of $\dot{\omega}_2$ from the equation (5.2)

$$\ddot{\omega}_1 = \left[\frac{(I_3 - I_2)(I_1 - I_3)}{I_1 I_2}\omega_3^2\right]\omega_1$$
$$\ddot{\omega}_1 = k^2\omega_1 \qquad k^2 = Constant$$

As ω_3^2 and $I_1 I_2$ are positive constant, the nature of the solution is decided by the product $(I_3 - I_2)(I_1 - I_3)$. If $I_3 > I_1$ and $I_3 > I_2$ or $I_3 < I_1$ and $I_3 < I_2$, the equation reduces to

$$\ddot{\omega}_1 = -k^2\omega_1$$

and the solution for w_1 will be oscillatory.
On the other hand, if $I_1 > I_3 > I_2$ or $I_1 < I_3 < I_2$, the equation becomes

$$\ddot{\omega}_1 = k^2\omega_1$$

the solution will be exponentially increasing with time. Similar arguments hold good for w_2 also. Hence, the rotation will be oscillatory if I_3 is either the largest or the smallest of the three principal moments of inertia.

Problem: 5.2- Calculate magnitude and direction of Coriolis acceleration of a rocket moving with a velocity of $2km/s$ at $60°$ south latitude.

Solution

For body moving in verticle direction, Coriolis force is,

$$\vec{F} = -2m\omega_y \dot{z}\hat{i}$$

For a rocket moving vertically upward at $60°$ south latitude

$$\vec{F} = -2m \times -\omega\cos 60° v\hat{i} = 2m\omega\cos 60° v\hat{i}$$

Magnitude of Coriolis acceleration is,

$$a_{\text{cor}} = 2\omega v\cos 60° = 2 \times \frac{2\pi}{60 \times 60 \times 24} \times 2 \times \cos 60°$$
$$\Rightarrow a_{\text{cor}} = 14.58 \times 10^{-5}\, m/s^2$$

Direction of Coriolis acceleration is towards east.

Problem: 5.3- The trace of a tensor is defined as the sum of the diagonal elements:

$$tr\{I\} \equiv \sum_k l_{kk}$$

Show, by performing a similarity transformation, that the trace is an invariant quantity. In other words, show that

$$tr\{I\} = tr\{I'\}$$

Where $\{I\}$ is the tensor in one coordinate system and $\{I'\}$ is the tensor in a coordinate system rotated with respect to the first system.

Solution

By definition,

$$I'_{ij} = \sum_{k,l} \lambda_{ik} I_{kl} \lambda_{lj}^{-1}$$

Then;

$$\begin{aligned} tr\{I\} &= \sum_i I'_{ii} = \sum_i \sum_{k,l} \lambda_{ik} I_{kl} \lambda_{li}^{-1} \\ &= \sum_{k,l} I_{kl} \sum_i \lambda_{li}^{-1} \lambda_{ik} \\ &= \sum_{k,l} I_{kl} \delta_{lk} = \sum_k I_{kk} \\ tr\{I\} &= tr\{I'\}, \qquad \text{As required.} \end{aligned}$$

Problem: 5.4- Calculate the moment of inertia I_1,I_2 and I_3 for a homogenous sphere of radius R and mass M.
Choose the origin at the center of the sphere.

Solution

Relation between cartesian and spherical coordinates is,

$$x = r\sin\theta\cos\phi, y = r\sin\theta\cos\phi, z = r\cos\theta$$

By definition of the moment of inertia,

$$I_{ij} = \int \rho(r)\left[\delta_{ij}\sum_k x_k^2 - x_i x_j\right] dv$$

Now

$$\begin{aligned}
I_{33} &= \rho\int (r^2 - z^2)dv \\
&= \rho\int (r^2 - r^2\cos^2\theta)r^2 dr d(\cos\theta)d\phi \\
I_{33} &= \rho\int_0^R r^4 dr \int_{-1}^{+1}(1-\cos^2\theta)d(\cos\theta)\int_0^{2\pi} d\phi \\
&= 2\pi\rho\frac{R^5}{5}\cdot\frac{4}{3}
\end{aligned}$$

The mass of sphere is

$$M = \frac{4\pi}{3}\rho R^3$$

Therefore,

$$I_{33} = \frac{2}{5}MR^2$$

Since the sphere is symmetrical around the origin, the diagonal elements of inertia are equal;

$$I_{11} = I_{22} = I_{33} = \frac{2}{5}MR^2 \longrightarrow (a)$$

A typical off-diagonal element is

$$I_{12} = \rho \int (-xy)dv$$
$$= -\rho \int r^2 \sin^2\theta \sin\phi \cos\phi r^2 dr d(\cos\theta) d\phi = 0$$

Therefore the eigen value equation is

$$\begin{vmatrix} I_{11} - I & 0 & 0 \\ 0 & I_{22} - I & 0 \\ 0 & 0 & I_{33} - I \end{vmatrix} = 0 \longrightarrow (b)$$

From (a) and (b), we have

$$I_1 = I_2 = I_3 = \frac{2}{5}MR^2$$

Problem: 5.5- Calculate the moments of inertia I_1,I_2 and I_3 for a homogenous ellipsoid of mass M with axes' length $2a > 2b > 2c$.

Solution

The equation of an ellipsoid is

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} = 1$$

It can be written in simple form if we make the following substitutions:

$$x_1 = au, \qquad x_2 = bv, \qquad x_3 = cw$$

The equation of ellipsoid reduces to

$$u^2 + v^2 + w^2 = 1$$

This is the equation of a sphere in the (u, v, w) system.
Volume of the ellipsoid is

$$V = \frac{4}{3}\pi abc$$

The rotational inertia with respect to the x_3-axis passing through the center of mass of the ellipsoid (we assume the ellipsoid to be homogenous), is given by

$$\begin{aligned} I_3 &= \frac{M}{V}\int (x_1^2 + x_2^2)dv \\ &= \frac{M}{V}abc\int (a^2u^2 + b^2v^2)d\tau \end{aligned}$$

Where $d\tau$ being volume element in (u, v, w) system. In order to evaluate this integral, consider the following equivalent integral in which $z = r\cos\theta$:

$$\begin{aligned} \int a^2z^2dv &= \int a^2z^2(rdrr\sin\theta d\theta d\phi) \\ &= a^2\int_0^{2\pi} d\phi \int_0^{\pi} \cos^2\theta \sin\theta d\theta \int_0^{R=1} r^4dr \\ &= a^2 \times 2\pi \times \frac{2}{3} \times \frac{1}{5} = \frac{4\pi a^2}{15} \end{aligned}$$

So

$$\begin{aligned} \int (a^2u^2 + b^2v^2)d\tau &= \frac{4\pi}{15}(a^2 + b^2) \\ and \quad I_3 &= \frac{1}{5}M(a^2 + b^2) \end{aligned}$$

Similarly the other moments of inertia are,

$$I_1 = \frac{1}{5}M(b^2 + c^2), \qquad I_2 = \frac{1}{5}M(a^2 + c^2)$$

Chapter 6

Hamilton's Equations of Motion

SOLVED PROBLEMS

Problem: 6.1- A system of two degrees of freedom is described by the Hamiltonian $H = q_1 p_1 - q_2 p_2 - a q_1^2 + b q_2^2$. Show that $F_1 = \frac{p_1 - a q_1}{q_2}$ and $F_2 = q_1 q_2$ are constants of motion.

Solution

$$H = q_1 p_1 - q_2 p_2 - a q_1^2 + b q_2^2 \tag{6.1}$$

$$F_1 = \frac{p_1 - a q_1}{q_2} \tag{6.2}$$

And,

$$F_2 = q_1 q_2 \tag{6.3}$$

The equations of motion for F_1 and F_2 are

$$\frac{dF_1}{dt} = [F_1, H] + \frac{\partial F_1}{\partial t}$$

$$\frac{dF_1}{dt} = [F_1, H] \qquad \because \frac{\partial F_1}{\partial t} = 0 \tag{6.4}$$

And,

$$\frac{dF_2}{dt} = [F_2, H] + \frac{\partial F_2}{\partial t}$$
$$\frac{dF_2}{dt} = [F_2, H] \qquad \because \frac{\partial F_2}{\partial t} = 0 \tag{6.5}$$

Since,

$$[F_1, H] = \sum_j \left[\frac{\partial F_1}{\partial q_j}.\frac{\partial H}{\partial p_j} - \frac{\partial F_1}{\partial p_j}.\frac{\partial H}{\partial q_j}\right]$$
$$\text{or} \quad [F_1, H] = \frac{\partial F_1}{\partial q_1}.\frac{\partial H}{\partial p_1} - \frac{\partial F_1}{\partial p_1}.\frac{\partial H}{\partial q_1} + \frac{\partial F_1}{\partial q_2}.\frac{\partial H}{\partial p_2} - \frac{\partial F_1}{\partial p_2}.\frac{\partial H}{\partial q_2} \tag{6.6}$$

Now,

$$\frac{\partial F_1}{\partial q_1} = \frac{\partial}{\partial q_1}\left[\frac{p_1 - aq_1}{q_2}\right]$$
$$\frac{\partial F_1}{\partial q_1} = \frac{1}{q_2}(0 - a)$$
$$\frac{\partial F_1}{\partial q_1} = -\frac{a}{q_2}$$

also,

$$\frac{\partial F_1}{\partial q_2} = -\frac{(p_1 - aq_1)}{q_2^2}$$

$$\frac{\partial F_1}{\partial p_1} = \frac{1}{q_2}; \quad \text{and} \quad \frac{\partial F_1}{\partial p_2} = 0$$
$$\frac{\partial H}{\partial q_1} = p_1 - 2aq_1; \quad \text{and} \quad \frac{\partial H}{\partial q_2} = -p_2 + 2bq_2$$
$$\frac{\partial H}{\partial p_1} = q_1; \quad \text{and} \quad \frac{\partial H}{\partial p_2} = -q_2$$

Substituting these values in Eq.(6.6) gives

$$[F_1, H] = -\frac{a}{q_2}.q_1 - \frac{1}{q_2}(p_1 - 2aq_1) + \left(\frac{-p_1 + aq_1)}{q_2^2}\right)(-q_2) - (0)(-p_2 + 2bq_2)$$
$$[F_1, H] = -\frac{aq_1}{q_2} - \frac{p_1}{q_2} + \frac{2aq_1}{q_2} + \frac{p_1}{q_1} - \frac{aq_1}{q_2} + 0$$
$$[F_1, H] = -\frac{2aq_1}{q_2} - \frac{p_1}{q_2} + \frac{2aq_1}{q_2} + \frac{p_1}{q_1} = 0 \tag{6.7}$$

From Eq.(6.4), $\frac{dF_1}{dt} = [F_1, H] = 0$. Hence F_1 is a constant of motion. Now, $F_2 = q_1q_2$, we have

$$\begin{aligned}
[F_2, H] &= \left[q_1q_2, q_1p_1 - q_2p_2 - aq_1^2 + bq_2^2\right] \\
[F_2, H] &= [q_1q_2, q_1p_1] - [q_1q_2, q_2p_2] - [q_1q_2, aq_1^2] + [q_1q_2, bq_2^2] \\
[F_2, H] &= q_1[q_2, p_1] - q_2[q_1, p_2] - aq_1[q_2, q_1] + bq_2[q_1, q_2] \\
[F_2, H] &= q_1[0] - q_2[0] - aq_1[0] + bq_2[0] = 0 \qquad (6.8)
\end{aligned}$$

From Eq.(6.5), $\frac{dF_2}{dt} = [F_2, H] = 0$. Hence F_2 is a constant of motion.

Problem: 6.2- Using the fundamental Poisson brackets find values of α and β for which the equation $Q = q^\alpha \cos\beta p$, and $P = q^\alpha \sin\beta p$ represent a canonical transformation. Also find a generating function F_3 for the transformation for some values of α & β.

Solution

$$Q = q^\alpha \cos\beta p \qquad (6.9)$$

and

$$P = q^\alpha \sin\beta p \qquad (6.10)$$

Now, from Eq.(6.9), we have

$$\frac{\partial Q}{\partial q} = \alpha q^{\alpha-1}\cos\beta p \quad \text{and} \quad \frac{\partial Q}{\partial p} = -q^\alpha \beta \sin\beta p$$

And, now from Eq.(6.10), we have

$$\frac{\partial P}{\partial q} = \alpha q^{\alpha-1}\sin\beta p \quad \text{and} \quad \frac{\partial P}{\partial p} = q^\alpha \beta \cos\beta p$$

For canonical transformation $[Q, P]_{q,p} = 1$, so

$$
\begin{aligned}
\frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} &= 1 \\
\alpha q^{\alpha-1}\cos\beta p . q^{\alpha}\beta\cos\beta p - \left[-q^{\alpha}\beta\sin\beta p\right] . \alpha q^{\alpha-1}\sin\beta p &= 1 \\
\alpha q^{\alpha-1}\cos\beta p . q^{\alpha}\beta\cos\beta p + q^{\alpha}\beta\sin\beta p . \alpha q^{\alpha-1}\sin\beta p &= 1 \\
\alpha\beta q^{2\alpha-1}\cos^2\beta p + \alpha\beta q^{2\alpha-1}\sin^2\beta p &= 1 \\
\alpha\beta q^{2\alpha-1}\left[\cos^2\beta p + \sin^2\beta p\right] &= 1 \\
\alpha\beta q^{2\alpha-1}(1) &= 1 \qquad \because \cos^2\beta p + \sin^2\beta p = 1 \\
\alpha\beta q^{2\alpha-1} &= 1 \\
\implies \alpha\beta &= 1 \quad \text{and} \quad q^{2\alpha-1} = q^0 \\
\implies 2\alpha - 1 &= 0 \\
\alpha &= \frac{1}{2}
\end{aligned}
$$

So, $\beta = \frac{1}{\alpha} = \frac{1}{1/2} = 2$, thus for $\alpha = \frac{1}{2}$ and $\beta = 2$, the transformation is canonical with transformation equation

$$Q = \sqrt{q}\cos 2p \tag{6.11}$$

$$P = \sqrt{q}\sin 2p \tag{6.12}$$

Now,

$$
\begin{aligned}
pdq - PdQ &= pdq - \sqrt{q}\sin 2p\left[\sqrt{q}\left(-2\sin 2pdp\right) + \cos 2p\frac{1}{2}.\frac{1}{\sqrt{2}}dq\right] \\
pdq - PdQ &= pdq + 2q\sin^2 2pdp - \frac{1}{2}\sin 2p\cos 2pdq \\
pdq - PdQ &= pdq + q(1 - \cos 4p)dp - \frac{1}{4}\sin 4pdq \\
pdq - PdQ &= pdq + qdp - q\cos 4pdp - \frac{1}{4}\sin 4pdq \\
pdq - PdQ &= (pdq + qdp) - \frac{1}{4}\left(4q\cos 4pdp + \sin 4pdq\right) \\
pdq - PdQ &= d(pq) - \frac{1}{4}d\left(q\sin 4p\right) \\
pdq - PdQ &= d\left(pq - \frac{1}{4}q\sin 4p\right) \\
pdq - PdQ &= dF_1
\end{aligned}
$$

Hence the generating function is

$$dF_1 = d\left(pq - \frac{1}{4}q\sin 4p\right)$$
$$F_1 = pq - \frac{1}{4}q\sin 4p$$

$$\because \quad F_3(p,Q) = F_1(q,Q) - q\frac{\partial F_1}{\partial q}$$
$$F_3(p,Q) = pq - \frac{1}{4}q\sin 4p - pq \qquad p = \frac{\partial F_1}{\partial q}$$
$$F_3(p,Q) = -\frac{1}{4}q\sin 4p \tag{6.13}$$

Now, from Eq.(6.11), we have

$$Q = \sqrt{q}\cos 2p$$
$$\Longrightarrow \quad \sqrt{q} = Q\sec 2p$$
$$q = Q^2\sec^2 2p \tag{6.14}$$

Substituting Eq.(6.14) into Eq.(6.13) gives

$$F_3(p,Q) = -\frac{1}{4}(Q^2\sec^2 2p)\sin 4p$$
$$F_3(p,Q) = -\frac{1}{4}\frac{Q^2}{\cos^2 2p}.2\sin 2p\cos 2p$$
$$F_3(p,Q) = -\frac{1}{2}\frac{Q^2}{\cos 2p}\sin 2p$$
$$F_3(p,Q) = -\frac{Q^2}{2}\frac{\sin 2p}{\cos 2p}$$
$$F_3(p,Q) = -\frac{Q^2}{2}\tan 2p$$

Problem: 6.3- Show directly that for a system of one degree of freedom, the transformation $Q = \tan^{-1}\left(\frac{\alpha q}{p}\right)$ and $P = \frac{\alpha q^2}{2}\left(1 + \frac{p^2}{\alpha^2 q^2}\right)$ is canonical, where α is a constant.

Solution

$$Q = \tan^{-1}\left(\frac{\alpha q}{p}\right) \tag{6.15}$$

And,

$$P = \frac{\alpha q^2}{2}\left(1 + \frac{p^2}{\alpha^2 q^2}\right)$$
$$P = \frac{\alpha q^2}{2} + \frac{p^2}{2\alpha} \tag{6.16}$$

If above given transformation is canonical, then $[Q, P] = 1$.

$$[Q, P] = \frac{\partial Q}{\partial q}.\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}.\frac{\partial P}{\partial q} \tag{6.17}$$

Now, using Eq.(6.15), we get

$$\frac{\partial Q}{\partial q} = \frac{\partial}{\partial q}\left[\tan^{-1}\left(\frac{\alpha q}{p}\right)\right]$$
$$\frac{\partial Q}{\partial q} = \frac{1}{1 + \left(\frac{\alpha q}{p}\right)^2}.\frac{\alpha}{p}$$

also,

$$\frac{\partial Q}{\partial p} = \frac{\partial}{\partial p}\left[\tan^{-1}\left(\frac{\alpha q}{p}\right)\right]$$
$$\frac{\partial Q}{\partial p} = \frac{1}{1 + \left(\frac{\alpha q}{p}\right)^2}.(\alpha q).\left(-\frac{1}{p^2}\right)$$
$$\frac{\partial Q}{\partial p} = -\frac{\alpha q}{p^2}.\frac{1}{1 + \left(\frac{\alpha q}{p}\right)^2}$$

Now, using Eq.(6.16), we get

$$\frac{\partial P}{\partial q} = \frac{\partial}{\partial q}\left[\frac{\alpha q^2}{2} + \frac{p^2}{2\alpha}\right]$$
$$\frac{\partial P}{\partial q} = \alpha q$$

also,

$$\frac{\partial P}{\partial p} = \frac{\partial}{\partial p}\left[\frac{\alpha q^2}{2} + \frac{p^2}{2\alpha}\right]$$
$$\frac{\partial P}{\partial p} = \frac{p}{\alpha}$$

Substituting values in Eq.(6.17), we get

$$[Q,P] = \frac{1}{1+\left(\frac{\alpha q}{p}\right)^2}.\frac{\alpha}{p}.\frac{p}{\alpha} - \left(-\frac{\alpha q}{p^2}\right).\frac{1}{1+\left(\frac{\alpha q}{p}\right)^2}.\alpha q$$
$$[Q,P] = \frac{1}{1+\left(\frac{\alpha q}{p}\right)^2} + \left(\frac{\alpha q}{p}\right)^2.\frac{1}{1+\left(\frac{\alpha q}{p}\right)^2}$$
$$[Q,P] = \frac{1}{1+\left(\frac{\alpha q}{p}\right)^2}\left[1+\left(\frac{\alpha q}{p}\right)^2\right]$$
$$[Q,P] = 1$$

Hence given transformation is canonical.

Problem: 6.4- Consider a function $f(q,p)$ of the coordinates q and p. Use Hamilton's equations to show that the time derivative of f can be written as

$$\frac{df}{dt} = \frac{\partial f}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial H}{\partial q}$$

Solution

From $f = f(q,p)$, we have

$$\frac{df}{dt} = \frac{\partial f}{\partial p}\dot{q} + \frac{\partial f}{\partial q}\dot{p}$$

Now by using Hamilton's equations

$$\frac{df}{dt} = \frac{\partial f}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial H}{\partial q}, \quad \text{As required}$$

Problem: 6.5- Set up Hamilton ins spherical polar coordinates.

Solution

Velocity of a particle in spherical polar coordinates is

$$\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r\dot{\phi}\sin\theta\hat{\phi} \quad \Rightarrow \; v^2 = \dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2\sin^2\theta$$

Kinetic energy is,

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2\sin^2\theta)$$

Lagrangian is,

$$L = T - V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2\sin^2\theta) - V(r,\theta,\phi)$$

Conjugate momentum to coordinate r is,

$$p_r = \frac{\partial L}{\partial \dot{r}} = \frac{1}{2}m(2\dot{r}) = m\dot{r} \qquad \Rightarrow \dot{r} = \frac{p_r}{m}$$

Conjugate momentum to coordinate θ is,

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = \frac{1}{2}mr^2(2\dot{\theta}) = mr^2\dot{\theta} \qquad \Rightarrow \dot{\theta} = \frac{p_\theta}{mr^2}$$

Conjugate momentum to coordinate ϕ is,

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = \frac{1}{2}mr^2(2\dot{\phi}\sin^2\theta) = mr^2\dot{\phi}\sin^2\theta \qquad \Rightarrow \dot{\phi} = \frac{p_\phi}{mr^2\sin^2\theta}$$

Hamiltonian of a system is,

$$H = T + V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2\sin^2\theta) + V(r,\theta,\phi)$$

$$\Rightarrow \qquad H = \frac{1}{2}m\left\{\frac{p_r^2}{m^2} + r^2\frac{p_\theta^2}{m^2r^4} + r^2\sin^2\theta\left(\frac{p_\phi^2}{m^2r^4\sin^4\theta}\right)\right\} + V(r,\theta,\phi)$$

$$\Rightarrow \qquad H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{p_\phi^2}{2mr^2\sin^2\theta} + V(r,\theta,\phi) \qquad \text{As required}$$

Chapter 7

Canonical Transformations

SOLVED PROBLEMS

Problem: 7.1- Find the Poisson bracket of $[L_x, L_y]$, where L_x and L_y are angular momentum components.

Solution

Angular momentum $\quad L = r \times P$

$$L_x = yp_z - zp_y \qquad L_y = zp_x - xp_z \qquad L_z = xp_y - yp_x$$

$$\begin{aligned}[L_x, L_y] &= [yp_z - zp_y, zp_x - xp_z] \\ &= [yp_z, zp_x] - [yp_z, xp_z] - [zp_y, zp_x] + [zp_y, xp_z]\end{aligned}$$

Consider the bracket $[yp_z, xp_z]$.

$$[yp_z, xp_z] = [y, x]p_z p_z + y[p_z, x]p_z + x[y, p_z]p_z + xy[p_z, p_z] = 0$$

Since all the fundamental brackets involved are zero. In the same way $[zp_y, zp_x] = 0$. Next we shall consider the Poison bracket $[yp_z, zp_x]$.

$$\begin{aligned}[yp_z, zp_x] &= [y, z]p_z p_x + y[p_z, z]p_x + z[y, p_x]p_z + zy[p_z, p_x] \\ &= 0 + y(-1)p_x + 0 + 0 = -yp_x\end{aligned}$$

In the same way

$$[zp_y, xp_z] = x(+1)p_y = xp_y$$

Substituting all the brackets

$$[L_x, L_y] = xp_y - yp_x = L_z$$

Proceeding on the same lines, we can show that

$$[L_y, L_z] = L_x \quad \text{and} \quad [L_z, L_x] = L_y$$

Note: In general, $[L_i, L_j] = L_k$, where i, j and k are taken in cyclic order. Let us introduce a symbol ϵ_{ijk} with following meaning:

1. $\epsilon_{ijk} = 0$, if two indices are equal.
   $\epsilon_{iii} = \epsilon_{iik} = \epsilon_{iji} = 0$
2. $\epsilon_{ijk} = 1$, if i, j, k are distinct and in cyclic order.
   $\epsilon_{ijk} = \epsilon_{jki} = \epsilon_{kij} = 1$
3. $\epsilon_{ijk} = -1$, if i, j, k are distinct and not in cyclic order.
   $\epsilon_{ikj} = \epsilon_{jik} = \epsilon_{kji} = -1$

The implication of the above result is that no two components of angular momentum can simultaneously act as conjugate momenta, since conjugate momenta must obey the relation $[p_i, p_j] = 0$. Only angular momentum component can be chosen as a generalized coordinate in any particular system of reference.

Problem: 7.2- Show directly that the transformation $Q = \log\left(\frac{1}{q}\sin p\right); P = q\cot p$ is canonical.

Solution

If transformation is canonical, then $[Q, P] = 1$.

$$1 = \frac{\partial Q}{\partial q}.\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}.\frac{\partial P}{\partial q} \tag{7.1}$$

Now,

$$\frac{\partial Q}{\partial q} = \frac{\partial}{\partial q}\left[\log\left(\frac{1}{q}\sin p\right)\right] = \frac{1}{\frac{1}{q}\sin p}.\sin p\left(-\frac{1}{q^2}\right) = -\frac{1}{q}$$

Also,

$$\frac{\partial Q}{\partial p} = \frac{\partial}{\partial p}\left[\log\left(\frac{1}{q}\sin p\right)\right] = \frac{1}{\frac{1}{q}\sin p}.\frac{1}{q}\cos p = \cot p$$

And,

$$\frac{\partial P}{\partial q} = \cot p$$

Also,

$$\frac{\partial P}{\partial p} = q(-\csc^2 p) = -q\csc^2 p$$

Substituting these values in Eq.(7.1), we get

$$\begin{aligned} 1 &= -\frac{1}{q}(-q\csc^2 p) - (\cot p)(\cot p) \\ 1 &= \csc^2 p - \cot^2 p \\ 1 &= 1 \qquad \text{Proved.} \end{aligned}$$

Problem: 7.3- The transformation equations between two sets of coordinates are

$$Q = \log\left(1 + \sqrt{q}\cos p\right) \tag{7.2}$$

and

$$P = 2\left(1 + \sqrt{q}\cos p\right)\sqrt{q}\sin p \tag{7.3}$$

(a) Show directly from these transformations that Q, P are canonical variables if q and p are.

(b) Show that the function that generates this transformation $F_3 = -\left(e^Q - 1\right)^2 \tan p$.

Solution

$$\begin{aligned} pdq - PdQ &= pdq - 2\left(1 + \sqrt{q}\cos p\right)\sqrt{q}\sin p.d\left[\log\left(1 + \sqrt{q}\cos p\right)\right] \\ pdq - PdQ &= pdq - 2\left(1 + \sqrt{q}\cos p\right)\sqrt{q}\sin p\frac{1}{1 + \sqrt{q}\cos p}.d\left[1 + \sqrt{q}\cos p\right] \\ pdq - PdQ &= pdq - 2\sqrt{q}\sin p.d\left[1 + \sqrt{q}\cos p\right] \end{aligned}$$

$$\begin{aligned}
pdq - PdQ &= pdq - 2\sqrt{q}\sin p\left[-\sqrt{q}\sin pdp + \frac{1}{2\sqrt{q}}\cos pdq\right]\\
pdq - PdQ &= pdq + 2q\sin^2 pdp - \sin p\cos pdq\\
pdq - PdQ &= pdq + q(1-\cos 2p)dp + \left(-\frac{1}{2}\sin 2p\right)dq \quad \because 2\sin 2p = 1-\cos 2p\\
\text{or}\quad pdq - PdQ &= pdq + qdp - q\cos 2pdp + \left(-\frac{1}{2}\sin 2p\right)dq\\
pdq - PdQ &= d(pq) - \frac{1}{2}d(q\sin 2p)\\
pdq - PdQ &= d\left(pq - \frac{1}{2}q\sin 2p\right)\\
pdq - PdQ &= dF_1 = \text{ exact differential}
\end{aligned}$$

Hence given transformation is canonical if P, Q are canonical variables. Now we have

$$\begin{aligned}
dF_1 &= d\left(pq - \frac{1}{2}q\sin 2p\right)\\
\text{or}\quad F_1(q,Q) &= pq - \frac{1}{2}q\sin 2p \qquad (7.4)
\end{aligned}$$

As,

$$\begin{aligned}
F_3(p,Q) &= F_1(q,Q) - q\frac{\partial F_1}{\partial q}\\
F_3(p,Q) &= F_1 - pq \qquad \because \frac{\partial F_1}{\partial q} = p \qquad (7.5)
\end{aligned}$$

Using Eq.(7.4) into Eq.(7.5) gives

$$\begin{aligned}
F_3(p,Q) &= pq - \frac{1}{2}q\sin 2p - pq\\
F_3(p,Q) &= -\frac{q}{2}\sin 2p \qquad (7.6)
\end{aligned}$$

From Eq.(7.2), we have

$$
\begin{aligned}
Q &= \log\left(1+\sqrt{q}\cos p\right)\\
\text{or}\quad e^{Q} &= 1+\sqrt{q}\cos p\\
\implies \quad \sqrt{q}\cos p &= e^{Q}-1\\
\sqrt{q} &= \frac{e^{Q}-1}{\cos p}\\
q &= \frac{(e^{Q}-1)^2}{\cos^2 p} \qquad (7.7)
\end{aligned}
$$

Substituting Eq.(7.7) into Eq.(7.6), we get

$$
\begin{aligned}
F_3(p,Q) &= -\frac{\frac{(e^{Q}-1)^2}{\cos^2 p}}{2}2\sin p\cos p \qquad \because \sin 2p = 2\sin p\cos p\\
F_3(p,Q) &= -\frac{(e^{Q}-1)^2}{2\cos^2 p}2\sin p\cos p\\
F_3(p,Q) &= -\frac{(e^{Q}-1)^2}{\cos p}\sin p\\
\text{or}\quad F_3(p,Q) &= -(e^{Q}-1)^2\frac{\sin p}{\cos p}\\
F_3(p,Q) &= -(e^{Q}-1)^2\tan p \qquad \text{Proved.}
\end{aligned}
$$

Problem: 7.4- One of the attempts at combining the two sets of Hamilton's equation in to one tries to take q and p as forming a complex quantity. Show directly that for a system of one degree of freedom the transformation $Q = q+ip, P = Q$ is not canonical if the Hamiltonian is left unaltered. Can you find another set of coordinates Q', P' that are related to Q, P by a change of scale only and that are canonical?

Solution

Given that

$$Q = q+ip \qquad (7.8)$$

and

$$P = Q^* = q-ip \qquad (7.9)$$

Let us generalize the given transformation a little;

$$Q = \alpha(q+ip) \qquad (7.10)$$

and

$$P = \beta(q - ip) \tag{7.11}$$

If $\alpha = 1$, Eq.(7.10) reduces to Eq.(7.8) and for $\beta = 1$, Eq.(7.11) reduces to Eq.(7.9). Now from Eq.(7.10), we get

$$\frac{Q}{\alpha} = q + ip$$

and from Eq.(7.11), we have

$$\frac{Q}{\beta} = q - ip$$

Adding these two equations, we get

$$\begin{aligned} q + ip + q - ip &= \frac{Q}{\alpha} + \frac{Q}{\beta} \\ q + q &= \frac{Q}{\alpha} + \frac{Q}{\beta} \\ 2q &= \frac{Q}{\alpha} + \frac{Q}{\beta} \\ \text{or} \quad q &= \frac{1}{2}\left(\frac{Q}{\alpha} + \frac{Q}{\beta}\right) \end{aligned} \tag{7.12}$$

Also subtracting these two equations, we get

$$\begin{aligned} q + ip - q + ip &= \frac{Q}{\alpha} - \frac{Q}{\beta} \\ ip + ip &= \frac{Q}{\alpha} - \frac{Q}{\beta} \\ 2ip &= \frac{Q}{\alpha} - \frac{Q}{\beta} \\ p &= \frac{1}{2i}\left(\frac{Q}{\alpha} - \frac{Q}{\beta}\right) \end{aligned} \tag{7.13}$$

The condition for canonical transformation requires that

$$[Q, P]_{q,p} = 1 \tag{7.14}$$

Now,

$$[Q, P]_{q,p} = \frac{\partial Q}{\partial q}.\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}.\frac{\partial P}{\partial q} \tag{7.15}$$

From Eq.(7.10), we get

$$\frac{\partial Q}{\partial q} = \alpha; \quad \text{and} \quad \frac{\partial Q}{\partial p} = i\alpha$$

and from Eq.(7.11), we get

$$\frac{\partial P}{\partial q} = \beta; \quad \text{and} \quad \frac{\partial P}{\partial p} = -i\beta$$

Substituting values in Eqs.(7.14) and (7.15) gives

$$\begin{aligned} \alpha(-i\beta) - i\alpha\beta &= 1 \\ -2\alpha\beta &= 1 \\ \alpha &= \frac{-1}{2i\beta} \end{aligned} \tag{7.16}$$

For reverse canonical transformation, required condition is $[q, p]_{Q,P} = 1$ or

$$1 = \frac{\partial q}{\partial Q} . \frac{\partial p}{\partial P} - \frac{\partial q}{\partial P} . \frac{\partial p}{\partial Q} \tag{7.17}$$

From Eqs.(7.12) and (7.13) required derivatives are

$$\begin{aligned} \frac{\partial q}{\partial Q} &= \frac{1}{2\alpha} \\ \frac{\partial q}{\partial P} &= \frac{1}{2\beta} \\ \frac{\partial p}{\partial Q} &= \frac{1}{2i\alpha} \\ \text{and} \quad \frac{\partial p}{\partial P} &= -\frac{1}{2i\beta} \end{aligned}$$

Substituting values in Eq.(7.17) gives

$$\begin{aligned} 1 &= \frac{1}{2\alpha}\left(-\frac{1}{2i\beta}\right) - \frac{1}{2\beta}\left(\frac{1}{2i\alpha}\right) \\ 1 &= -\frac{1}{4i\alpha\beta} - \frac{1}{4i\alpha\beta} \end{aligned}$$

$$1 = -\frac{1}{2i\alpha\beta}$$
$$\implies \alpha = \frac{-1}{2i\beta} \tag{7.18}$$

Thus for both transformations (forward and reverse)

$$\alpha = \frac{-1}{2i\beta} \tag{7.19}$$

For forward transformation if $\alpha = 1, \beta = 1$, which does not satisfy Eq.(7.19), it means that transformation given by Eqs.(7.8) and (7.9) is not canonical. However, if $\alpha = 1, \beta = -\frac{1}{2i}$, Eq.(7.19) is satisfied, hence the transformation $Q = q + ip$ and $P = -\frac{1}{2i}(q - ip) = -\frac{1}{2i}Q^*$ is canonical.

Problem: 7.5- Determine whether the transformation

1. $Q_1 = q_1 q_2$
2. $P_1 = \frac{p_1 - p_2}{q_2 - q_1} + 1$
3. $Q_2 = q_1 + q_2$
4. $P_2 = \frac{q_2 p_2 - q_1 p_1}{q_2 - q_1} - (q_2 + q_1)$

is canonical.

Solution

$$\begin{aligned}
p_1dq_1 - P_1dQ_1 + p_2dq_2 - P_2dQ_2 &= p_1dq_1 - \left[\frac{p_1 - p_2}{q_2 - q_1} + 1\right] d(q_1q_2) + p_2dq_2 \\
&\quad - \left[\frac{q_2p_2 - q_1p_1}{q_2 - q_1} - (q_2 + q_1)\right] d(q_1 + q_2) \\
p_1dq_1 - P_1dQ_1 + p_2dq_2 - P_2dQ_2 &= p_1dq_1 - \left[\frac{p_1 - p_2}{q_2 - q_1} + 1\right] (q_1dq_2 + q_2dq_1) + p_2dq_2 \\
&\quad - \left[\frac{q_2p_2 - q_1p_1}{q_2 - q_1} - (q_2 + q_1)\right] (dq_1 + dq_2) \\
p_1dq_1 - P_1dQ_1 + p_2dq_2 - P_2dQ_2 &= p_1dq_1 - \frac{p_1q_1}{q_2 - q_1}dq_2 + \frac{q_1p_2}{q_2 - q_1}dq_2 - q_1dq_2 \\
&\quad - \frac{q_2p_1}{q_2 - q_1}dq_1 - q_2dq_1 + p_2dq_2 - \frac{q_2p_2}{q_2 - q_1}dq_1
\end{aligned}$$

$$+\frac{q_1p_1}{q_2-q_1}dq_1+q_2dq_1+q_1dq_1-\frac{q_2p_2}{q_2-q_1}dq_2$$
$$+\frac{q_1p_1}{q_2-q_1}dq_2+q_2dq_2+q_1dq_2$$
$$p_1dq_1-P_1dQ_1+p_2dq_2-P_2dQ_2 = p_1dq_1+(q_1-q_2)\frac{p_2}{(q_2-q_1)}dq_2-(q_2-q_1)\frac{p_1}{(q_2-q_1)}dq_1$$
$$+p_2dq_2+q_1dq_1+q_2dq_2$$
$$p_1dq_1-P_1dQ_1+p_2dq_2-P_2dQ_2 = p_1dq_1-p_2dq_2-p_1dq_1+p_2dq_2+q_1dq_1+q_2dq_2$$
$$p_1dq_1-P_1dQ_1+p_2dq_2-P_2dQ_2 = q_1dq_1+q_2dq_2 \neq \text{exact differential}$$

so transformation is not canonical.

Problem: 7.6- Show by the use of Poisson brackets that for a one-dimensional harmonic oscillator; there is a constant of the motion u defined as:

$$u(q,p,t) = \ln(p+im\omega q)-i\omega t, \quad \omega = \sqrt{\frac{k}{m}}.$$

Solution

For a one-dimensional harmonic oscillator having coordinate q and momentum p, the kinetic and potential energies are given by.

$$T = \frac{1}{2}mv^2$$
$$T = \frac{p^2}{2m}$$

and,

$$V = \frac{kq^2}{2}$$
$$V = \frac{m\omega^2q^2}{2} \qquad \because \omega = \sqrt{\frac{k}{m}}$$

The Hamiltonian for one-dimensional harmonic oscillator is

$$\begin{aligned} H &= T + V \\ H &= \frac{p^2}{2m} + \frac{m\omega^2 q^2}{2} \\ H &= \frac{1}{2m}\left(p^2 + m^2\omega^2 q^2\right) \end{aligned} \tag{7.20}$$

Now,

$$\begin{aligned} \frac{\partial H}{\partial p} &= \frac{\partial}{\partial p}\left[\frac{1}{2m}\left(p^2 + m^2\omega^2 q^2\right)\right] \\ \frac{\partial H}{\partial p} &= \frac{1}{2m}(2p) \\ \frac{\partial H}{\partial p} &= \frac{p}{m} \end{aligned}$$

and

$$\begin{aligned} \frac{\partial H}{\partial q} &= \frac{\partial}{\partial q}\left[\frac{1}{2m}\left(p^2 + m^2\omega^2 q^2\right)\right] \\ \frac{\partial H}{\partial q} &= \frac{1}{2m}(0 + m^2\omega^2 2q) \\ \frac{\partial H}{\partial q} &= m\omega^2 q \end{aligned}$$

As,

$$u(q,p,t) = \ln(p + im\omega q) - i\omega t$$

So,

$$\frac{\partial u}{\partial p} = \frac{1}{p + im\omega q}$$

also,

$$\begin{aligned} \frac{\partial u}{\partial q} &= \frac{1}{p + im\omega q}(0 + im\omega) \\ \frac{\partial u}{\partial q} &= \frac{im\omega}{p + im\omega q} \end{aligned}$$

And,

$$\frac{\partial u}{\partial t} = -i\omega$$

The equation of motion for $u(q,p,t)$ is given by

$$\frac{du}{dt} = [u, H] + \frac{\partial u}{\partial t}$$
$$\implies \frac{du}{dt} = \frac{\partial u}{\partial q}.\frac{\partial H}{\partial p} - \frac{\partial u}{\partial p}.\frac{\partial H}{\partial q} + \frac{\partial u}{\partial t}$$
$$\frac{du}{dt} = \frac{im\omega}{p + im\omega q}\left(\frac{p}{m}\right) - \frac{1}{p + im\omega q}(m\omega^2 q) - i\omega$$
$$\frac{du}{dt} = \frac{i\omega p}{p + im\omega q} + \frac{i^2 m\omega^2 q}{p + im\omega q} - i\omega$$
$$\frac{du}{dt} = i\omega\left[\frac{p}{p + im\omega q} + \frac{im\omega q}{p + im\omega q}\right] - i\omega$$
$$\frac{du}{dt} = i\omega\left[\frac{p + im\omega q}{p + im\omega q}\right] - i\omega$$
$$\frac{du}{dt} = i\omega(1) - i\omega$$
$$\frac{du}{dt} = i\omega - i\omega$$
$$\frac{du}{dt} = 0$$

Hence u is a constant of the motion.

Problem: 7.7-

(a) For one dimensional system with the Hamiltonian $H = \frac{p^2}{2} - \frac{1}{2q^2}$, show that there is a constant of motion $D = \frac{pq}{2} - Ht$.

(b) As a generalization of part (a), for motion in plane with Hamiltonian $H = \left|\vec{p}\right|^n - ar^{-n}$, where $\vec{p}$ is the vector of the momenta conjugate to the Cartesian coordinates, show that there is a constant of the motion $D = \frac{\vec{p}.\vec{r}}{n} - Ht$.

(c) The transformation $Q = \lambda q, p = \lambda P$ is obviously canonical. However the same transformation with t time dilatation, $Q = \lambda q, p = \lambda P, t' = \lambda^2 t$ is not. Show that, however, the equations of motion for q and p for the Hamiltonian in part (a) are invariant under the transformation. The constant of motion D is said to be associated with this invariance.

Solution

(a) The equation of motion for the quantity D is given by:

$$\frac{dD}{dt} = [D, H] + \frac{\partial D}{\partial t} \tag{7.21}$$

And,

$$[D, H]_{q,p} = \frac{\partial D}{\partial q}.\frac{\partial H}{\partial p} - \frac{\partial D}{\partial p}.\frac{\partial H}{\partial q} \tag{7.22}$$

Since $H = \frac{p^2}{2} - \frac{1}{2q^2}$, so

$$\frac{\partial H}{\partial q} = -\frac{1}{2}\left(-\frac{2}{q^3}\right)$$
$$\frac{\partial H}{\partial q} = \frac{1}{q^3} \tag{7.23}$$

And,

$$\frac{\partial H}{\partial p} = \frac{1}{2}(2p)$$
$$\frac{\partial H}{\partial p} = p \tag{7.24}$$

As $D = \frac{pq}{2} - Ht$, so

$$\frac{\partial D}{\partial q} = \frac{p}{2} \tag{7.25}$$
$$\frac{\partial D}{\partial p} = \frac{q}{2} \tag{7.26}$$

and

$$\frac{\partial D}{\partial t} = -H \tag{7.27}$$

Substituting values from Eqs.(7.23), (7.24), (7.25) and (7.26) into Eq.(7.22) gives

$$[D, H] = \frac{p}{2}.p - \frac{q}{2}\left(\frac{1}{q^3}\right)$$
$$[D, H] = \frac{p^2}{2} - \frac{1}{2q^2} \tag{7.28}$$

Substituting Eqs.(7.27) and (7.28) into Eq.(7.21) gives

$$\frac{dD}{dt} = \frac{p^2}{2} - \frac{1}{2q^2} + (-H)$$
$$\frac{dD}{dt} = \frac{p^2}{2} - \frac{1}{2q^2} - \frac{p^2}{2} + \frac{1}{2q^2}$$
$$\frac{dD}{dt} = 0$$

As $\frac{dD}{dt} = 0 \implies D$ is a constant of the motion.

(b) Let

$$\vec{p} = p_1\hat{i} + p_2\hat{j}$$
$$\vec{r} = q_1\hat{i} + q_2\hat{j}$$

So,

$$\left|\vec{r}\right| = r = \sqrt{q_1^2 + q_2^2}$$

and

$$\left|\vec{p}\right| = p = \sqrt{p_1^2 + p_2^2}$$

Now as

$$H = \left|\vec{p}\right|^n - ar^{-n}$$
$$H = \left[\sqrt{p_1^2 + p_2^2}\right]^n - a\left[\sqrt{q_1^2 + q_2^2}\right]^{-n}$$
$$\text{or } H = \left[p_1^2 + p_2^2\right]^{n/2} - a\left[q_1^2 + q_2^2\right]^{-n/2}$$

Now,

$$\frac{\partial H}{\partial q_j} = -a\left[-\frac{n}{2}\left(q_1^2 + q_2^2\right)^{-\frac{n}{2}-1}.2q_j\right]$$
$$\frac{\partial H}{\partial q_j} = naq_j\left(q_1^2 + q_2^2\right)^{-\frac{n}{2}-1} \tag{7.29}$$

And

$$\frac{\partial H}{\partial p_j} = \frac{n}{2}\left(p_1^2 + p_2^2\right)^{\frac{n}{2}-1}.2p_j$$
$$\frac{\partial H}{\partial p_j} = np_j\left(p_1^2 + p_2^2\right)^{\frac{n}{2}-1} \tag{7.30}$$

Also,

$$\vec{p}\cdot\vec{r} = \left(p_1\hat{i} + p_2\hat{j}\right)\cdot\left(q_1\hat{i} + q_2\hat{j}\right)$$
$$\vec{p}\cdot\vec{r} = p_1q_1 + p_2q_2$$

Now,

$$\begin{aligned}
\frac{\partial \vec{p}\cdot\vec{r}}{\partial q_j} &= \frac{\partial}{\partial q_j}\left(p_1 q_1 + p_2 q_2\right) \\
\frac{\partial \vec{p}\cdot\vec{r}}{\partial q_j} &= p_j \qquad\qquad \because j \;=\; 1,2 \qquad (7.31)
\end{aligned}$$

And,

$$\begin{aligned}
\frac{\partial \vec{p}\cdot\vec{r}}{\partial p_j} &= \frac{\partial}{\partial q_j}\left(p_1 q_1 + p_2 q_2\right) \\
\frac{\partial \vec{p}\cdot\vec{r}}{\partial p_j} &= q_j \qquad\qquad \because j \;=\; 1,2 \qquad (7.32)
\end{aligned}$$

Now,

$$[\vec{p}\cdot\vec{r}, H]q,p \;=\; \sum j\left[\frac{\partial \vec{p}\cdot\vec{r}}{\partial q_j}.\frac{\partial H}{\partial p_j} - \frac{\partial \vec{p}\cdot\vec{r}}{\partial p_j}.\frac{\partial H}{\partial q_j}\right] \qquad (7.33)$$

Substituting values from Eqs.(7.29), (7.30), (7.31) and (7.32) in Eq.(7.33), we get

$$\begin{aligned}
[\vec{p}\cdot\vec{r}, H]_{q,p} &= \sum_j\left[p_j\left\{np_j\left(p_1^2+p_2^2\right)^{\frac{n}{2}-1}\right\} - q_j\left\{naq_j\left(q_1^2+q_2^2\right)^{-\frac{n}{2}-1}\right\}\right] \\
[\vec{p}\cdot\vec{r}, H]_{q,p} &= \sum_j\left[np_j^2\left(p_1^2+p_2^2\right)^{\frac{n}{2}-1} - naq_j^2\left(q_1^2+q_2^2\right)^{-\frac{n}{2}-1}\right] \\
[\vec{p}\cdot\vec{r}, H] &= np_1^2\left(p_1^2+p_2^2\right)^{\frac{n}{2}-1} - naq_1^2\left(q_1^2+q_2^2\right)^{-\frac{n}{2}-1} + np_2^2\left(p_1^2+p_2^2\right)^{\frac{n}{2}-1} \\
&\quad - naq_2^2\left(q_1^2+q_2^2\right)^{-\frac{n}{2}-1} \\
[\vec{p}\cdot\vec{r}, H] &= n\left(p_1^2+p_2^2\right)^{\frac{n}{2}-1}\left(p_1^2+p_2^2\right) - na\left(q_1^2+q_2^2\right)^{-\frac{n}{2}-1}\left(q_1^2+q_2^2\right) \\
[\vec{p}\cdot\vec{r}, H] &= n\left(p_1^2+p_2^2\right)^{\frac{n}{2}} - na\left(q_1^2+q_2^2\right)^{-\frac{n}{2}} \\
\Longrightarrow \quad \frac{1}{n}[\vec{p}\cdot\vec{r}, H] &= \left(p_1^2+p_2^2\right)^{\frac{n}{2}} - a\left(q_1^2+q_2^2\right)^{-\frac{n}{2}} \;=\; \left|\vec{p}\right|^n - ar^{-n} \;=\; H \qquad (7.34)
\end{aligned}$$

Also,

$$\begin{aligned}
D &= \frac{\vec{p}\cdot\vec{r}}{n} - Ht \\
\Longrightarrow \quad \frac{\partial D}{\partial t} &= -\,H \qquad (7.35)
\end{aligned}$$

Also,

$$
\begin{aligned}
[D,H] &= \left[\frac{\vec{p}\cdot\vec{r}}{n} - Ht, H\right] \\
[D,H] &= \left[\frac{\vec{p}\cdot\vec{r}}{n}, H\right] - [H,H] \\
[D,H] &= \frac{1}{n}[\vec{p}\cdot\vec{r}, H] - t[H,H] \\
[D,H] &= \frac{1}{n}[\vec{p}\cdot\vec{r}, H] - 0 \qquad \because [H,H] = 0 \\
[D,H] &= \frac{1}{n}[\vec{p}\cdot\vec{r}, H] \qquad (7.36)
\end{aligned}
$$

The equation of motion for D is given by;

$$
\frac{dD}{dt} = [D,H] + \frac{\partial D}{\partial t} \qquad (7.37)
$$

Substituting values from Eq.(7.35) and (7.36) into Eq.(7.37) gives

$$
\begin{aligned}
\frac{dD}{dt} &= \frac{1}{n}[\vec{p}\cdot\vec{r}, H] - H \\
\frac{dD}{dt} &= H - H \qquad \because \text{Using Eq.(7.34)} \\
\frac{dD}{dt} &= 0
\end{aligned}
$$

As, $\frac{dD}{dt} = 0$ so D is constant of the motion.

(c) Since

$$
Q = \lambda q \qquad (7.38)
$$

and

$$
\begin{aligned}
\lambda P &= p \\
P &= \frac{1}{\lambda} p \qquad (7.39)
\end{aligned}
$$

And,

$$
\begin{aligned}
t' &= \lambda^2 t \\
\implies \quad t &= \frac{t'}{\lambda^2} \qquad (7.40)
\end{aligned}
$$

By taking q, p and Q, P as functions of time t and t' respectively we can rewrite Eqs.(7.38) and (7.39) as follows;

$$\begin{aligned} Q(t') &= \lambda q(t) \\ \implies \quad Q(t') &= \lambda q\left(\frac{t'}{\lambda^2}\right) \qquad \because \text{Using Eq.(7.40)} \end{aligned} \tag{7.41}$$

and

$$\begin{aligned} P(t') &= \frac{1}{\lambda} p(t) \\ \implies \quad P(t') &= \frac{1}{\lambda} p\left(\frac{t'}{\lambda^2}\right) \qquad \because \text{Using Eq.(7.40)} \end{aligned} \tag{7.42}$$

As p and q are old parameters, they satisfy Hamilton's equation.

$$\dot{q} = \frac{\partial H}{\partial p}; \quad \text{and} \quad \dot{p} = -\frac{\partial H}{\partial q}$$

As, $H = \frac{p^2}{2} - \frac{1}{2q^2}$, so

$$\begin{aligned} \frac{\partial H}{\partial q} &= \frac{\partial}{\partial q}\left(\frac{p^2}{2} - \frac{1}{2q^2}\right) \\ \frac{\partial H}{\partial q} &= -\frac{1}{2}\left(-\frac{2}{q^3}\right) \\ \frac{\partial H}{\partial q} &= \frac{1}{q^3} \end{aligned}$$

And,

$$\begin{aligned} \frac{\partial H}{\partial p} &= \frac{\partial}{\partial p}\left(\frac{p^2}{2} - \frac{1}{2q^2}\right) \\ \frac{\partial H}{\partial p} &= \frac{1}{2}(2p) \\ \frac{\partial H}{\partial p} &= p \end{aligned}$$

So, we have

$$\dot{q} = p \tag{7.43}$$

and

$$\dot{p} = -\frac{1}{q^3} \tag{7.44}$$

Taking time derivative of Eq.(7.41) gives

$$\begin{aligned}
\frac{dQ(t')}{dt'} &= \frac{d}{dt'}\left[\lambda q\left(\frac{t'}{\lambda^2}\right)\right] \\
\dot{Q}(t') &= \lambda\frac{d}{dt}q\left(\frac{t'}{\lambda^2}\right).\frac{1}{\lambda^2} \\
\dot{Q}(t') &= \lambda\dot{q}\left(\frac{t'}{\lambda^2}\right).\frac{1}{\lambda^2} \\
\dot{Q}(t') &= \frac{1}{\lambda}\dot{q}\left(\frac{t'}{\lambda^2}\right) \\
\dot{Q}(t') &= \frac{1}{\lambda}p\left(\frac{t'}{\lambda^2}\right) && \because \text{Using Eq.(7.43)} \\
\Longrightarrow \quad \dot{Q}(t') &= P(t') && \because \text{Using Eq.(7.42)}
\end{aligned} \tag{7.45}$$

Similarly taking time derivative of Eq.(7.42) gives

$$\begin{aligned}
\frac{dP(t')}{dt'} &= \frac{d}{dt'}\left[\frac{1}{\lambda}p\left(\frac{t'}{\lambda^2}\right)\right] \\
\dot{P}(t') &= \frac{1}{\lambda}\frac{d}{dt}p\left(\frac{t'}{\lambda^2}\right).\frac{1}{\lambda^2} \\
\dot{P}(t') &= \frac{1}{\lambda}\dot{p}\left(\frac{t'}{\lambda^2}\right).\frac{1}{\lambda^2} \\
\dot{P}(t') &= \frac{1}{\lambda^3}\dot{p}\left(\frac{t'}{\lambda^2}\right) \\
\dot{P}(t') &= \frac{1}{\lambda^3}\left(-\frac{1}{q^3}\right)\left(\frac{t'}{\lambda^2}\right) && \because \text{Using Eq.(7.44)} \\
\dot{P}(t') &= -\frac{1}{\lambda^3 q^3\left(\frac{t'}{\lambda^2}\right)} \\
\dot{P}(t') &= -\frac{1}{Q^3(t')} && \because \text{Using Eq.(7.41)}
\end{aligned} \tag{7.46}$$

So from Eqs.(7.43), (7.44), (7.45) and (7.46) we can write as;

$$\dot{q} = p \quad ; \quad \dot{Q} = P$$
$$\text{and} \quad \dot{p} = -\frac{1}{q^3} \quad ; \quad \dot{P} = -\frac{1}{Q^3}$$

Hence transformation is invariant as both set of equations are the same.

Problem: 7.8-

(a) Prove that the Poisson bracket of two constants of the motion is itself a constant of the motion even when the constants of motion depend on time explicitly.

(b) Show that if the Hamiltonian and a quantity F are constants of the motion, then the nth partial derivative of F with respect to t must also be a constants of the motion.

(c) As an illustration of this result, consider the uniform motion of a free particle of mass m. The Hamiltonian is certainly conserved and there exists a constant of the motion, agrees with $[H, F]$.

Solution

(a)

$$\begin{aligned}
\frac{du}{dt} &= [u, H] + \frac{\partial u}{\partial t} \\
0 &= [u, H] + \frac{\partial u}{\partial t} \qquad \because \frac{du}{dt} = 0 \\
-[u, H] &= \frac{\partial u}{\partial t} \\
\text{or} \quad [H, u] &= \frac{\partial u}{\partial t} \qquad (7.47)
\end{aligned}$$

And

$$\begin{aligned}
\frac{dv}{dt} &= [v, H] + \frac{\partial v}{\partial t} \\
0 &= [v, H] + \frac{\partial v}{\partial t} \qquad \because \frac{dv}{dt} = 0 \\
-[v, H] &= \frac{\partial v}{\partial t} \\
\text{or} \quad [H, v] &= \frac{\partial v}{\partial t} \qquad (7.48)
\end{aligned}$$

Since

$$\frac{du}{dt} = [u, H] + \frac{\partial u}{\partial t} \tag{7.49}$$

Now substituting u by $[u, v]$ in Eq.(7.49), gives

$$\begin{aligned}\frac{d[u,v]}{dt} &= [[u,v],H] + \frac{\partial [u,v]}{\partial t} \\ \frac{d[u,v]}{dt} &= [[u,v],H] + \left[\frac{\partial u}{\partial t}, v\right] + \left[u, \frac{\partial v}{\partial t}\right]\end{aligned} \tag{7.50}$$

Substituting Eqs.(7.47) and (7.48) in Eq.(7.50) gives

$$\begin{aligned}\frac{d[u,v]}{dt} &= [[u,v],H] + [-[u,H],v] + [u,-[v,H]] \\ \frac{d[u,v]}{dt} &= [[u,v],H] - [[u,H],v] - [u,[v,H]] \\ \because [u,H] &= -[H,u] \text{and } [u,[v,H]] = -[[v,H],u] \\ \frac{d[u,v]}{dt} &= [[u,v],H] + [[H,u],v] + [[v,H],u] \\ \text{or } \frac{d[u,v]}{dt} &= [[u,v],H] + [[v,H],u] + [[H,u],v] \\ 0 &= [[u,v],H] + [[v,H],u] + [[H,u],v] \quad \because \frac{d[u,v]}{dt} = 0\end{aligned}$$

Or

$$[[u,v],H] + [[v,H],u] + [[H,u] = 0$$

Hence we have $[u, v] =$ constant.

(b)

If F is a constant of motion, then $\frac{dF}{dt} = 0$, so that the equation of the motion for F will become

$$\begin{aligned}\frac{dF}{dt} &= [F,H] + \frac{\partial F}{\partial t} \\ \text{or } \ 0 &= [F,H] + \frac{\partial F}{\partial t} \\ \Longrightarrow \quad \frac{\partial F}{\partial t} &= -[F,H]\end{aligned} \tag{7.51}$$

As H is also a constant of the motion, then $\frac{dH}{dt} = 0$, so that the equation of the motion for H will become

$$\begin{aligned}\frac{dH}{dt} &= [H, H] + \frac{\partial H}{\partial t} \\ \text{or } 0 &= 0 + \frac{\partial H}{\partial t} & \because [H, H] \;=\; 0 \\ \implies \quad \frac{\partial H}{\partial t} &= 0 \end{aligned} \tag{7.52}$$

The equation of motion for $\frac{\partial^n F}{\partial t^n}$ is given as:

$$\frac{d}{dt}\left[\frac{\partial^n F}{\partial t^n}\right] \;=\; \left[\frac{\partial^n F}{\partial t^n}, H\right] + \frac{\partial}{\partial t}\left[\frac{\partial^n F}{\partial t^n}\right] \tag{7.53}$$

By taking the nth partial derivative of Eq.(7.51) gives

$$\begin{aligned}\frac{\partial^n}{\partial t^n}\left[\frac{\partial F}{\partial t}\right] &= -\frac{\partial^n}{\partial t^n}[F, H] \\ \frac{\partial^n}{\partial t^n}\left[\frac{\partial F}{\partial t}\right] &= -\left[\frac{\partial^n F}{\partial t^n}, H\right] - \left[F, \frac{\partial^n H}{\partial t^n}\right] \end{aligned} \tag{7.54}$$

As from Eq.(7.52), we have

$$\begin{aligned}\frac{\partial H}{\partial t} &= 0 \\ \implies \frac{\partial^2 H}{\partial t^2} &= 0 \\ \frac{\partial^3 H}{\partial t^3} &= 0 \\ \vdots &= \vdots \\ \frac{\partial^n H}{\partial t^n} &= 0 \end{aligned}$$

So that

$$\begin{aligned}\left[F, \frac{\partial^n H}{\partial t^n}\right] &= [F, 0] \\ \left[F, \frac{\partial^n H}{\partial t^n}\right] &= 0 \end{aligned}$$

Now, Eq.(7.54) can be written as

$$\frac{\partial^n}{\partial t^n}\left[\frac{\partial F}{\partial t}\right] = -\left[\frac{\partial^n F}{\partial t^n}, H\right] - 0$$

$$\frac{\partial^n}{\partial t^n}\left[\frac{\partial F}{\partial t}\right] = -\left[\frac{\partial^n F}{\partial t^n}, H\right]$$

$$\text{or } \frac{\partial}{\partial t}\left[\frac{\partial^n F}{\partial t^n}\right] = -\left[\frac{\partial^n F}{\partial t^n}, H\right] \tag{7.55}$$

Substituting Eq.(7.55) into Eq.(7.53) gives

$$\frac{d}{dt}\left[\frac{\partial^n F}{\partial t^n}\right] = \left[\frac{\partial^n F}{\partial t^n}, H\right] - \left[\frac{\partial^n F}{\partial t^n}, H\right]$$

$$\frac{d}{dt}\left[\frac{\partial^n F}{\partial t^n}\right] = 0$$

Hence the nth partial derivative of $F = \frac{\partial^n F}{\partial t^n}$ with t is also a constant.

(c)

$$F = x - \frac{pt}{m}$$

$$\Longrightarrow \frac{\partial F}{\partial t} = -\frac{p}{m} \tag{7.56}$$

Now, equation of motion for $\frac{\partial F}{\partial t}$ is given by

$$\frac{d}{dt}\left[\frac{\partial F}{\partial t}\right] = \left[\frac{\partial F}{\partial t}, H\right] + \frac{\partial}{\partial t}\left[\frac{\partial F}{\partial t}\right]$$

$$\frac{d}{dt}\left[\frac{\partial F}{\partial t}\right] = \left[\frac{\partial F}{\partial t}, H\right] + 0 \qquad \because \text{Using Eq.(7.56)}$$

$$\frac{d}{dt}\left[\frac{\partial F}{\partial t}\right] = \left[\frac{\partial F}{\partial t}, H\right] \tag{7.57}$$

Now,

$$\left[\frac{\partial F}{\partial t}, H\right] = \frac{\partial}{\partial q}\left(\frac{\partial F}{\partial t}\right).\frac{\partial H}{\partial p} - \frac{\partial}{\partial p}\left(\frac{\partial F}{\partial t}\right).\frac{\partial H}{\partial q} \tag{7.58}$$

Let Hamiltonian for a free particle is $H = \frac{p^2}{2m} + mgy$, for $q = x$, we have

$$\frac{\partial}{\partial q}\left(\frac{\partial F}{\partial t}\right) = 0; \quad \text{and} \quad \frac{\partial}{\partial p}\left(\frac{\partial F}{\partial t}\right) = -\frac{1}{m}$$

Also,

$$\frac{\partial H}{\partial q} = 0; \quad \text{and} \quad \frac{\partial H}{\partial p} = \frac{p}{m}$$

Substituting these values in Eq.(7.58) gives

$$\begin{aligned}
\left[\frac{\partial F}{\partial t}, H\right] &= (0).\left(\frac{p}{m}\right) - \left(-\frac{1}{m}\right)(0) \\
\left[\frac{\partial F}{\partial t}, H\right] &= 0 + 0 \\
\left[\frac{\partial F}{\partial t}, H\right] &= 0 \qquad (7.59)
\end{aligned}$$

Hence from Eqs.(7.57) and (7.59) one can write

$$\frac{d}{dt}\left[\frac{\partial F}{\partial t}\right] = 0$$

So that $\frac{\partial F}{\partial t}$ is a constant of motion.

Problem: 7.9- Show directly that the transformation

1. $Q_1 = q_1$
2. $P_1 = p_1 - 2p_2$
3. $Q_2 = p_2$
4. $P_2 = -2q_1 - q_2$

is canonical and find a generating function.

Solution

$$\begin{aligned}
-P_1dQ_1 - P_2dQ_2 + p_1dq_1 + p_2dq_2 &= -(p_1 - 2p_2)dq_1 - (-2q_1 + q_2)dp_2 + p_1dq_1 + p_2dq_2 \\
-P_1dQ_1 - P_2dQ_2 + p_1dq_1 + p_2dq_2 &= -p_1dq_1 + 2p_2dq_1 + 2q_1dp_2 - q_2dp_2 + p_1dq_1 + p_2dq_2 \\
-P_1dQ_1 - P_2dQ_2 + p_1dq_1 + p_2dq_2 &= 2d(q_1p_2) + d(2p_2q_2) \\
-P_1dQ_1 - P_2dQ_2 + p_1dq_1 + p_2dq_2 &= d(2q_1p_2 + q_2p_2) \\
-P_1dQ_1 - P_2dQ_2 + p_1dq_1 + p_2dq_2 &= \text{exact differential}
\end{aligned}$$

Hence the transformation is canonical. Now,

$$dF_1 = d(2q_1p_2 + q_2p_2)$$
$$F_1 = 2q_1p_2 + q_2p_2$$
$$F_1 = p_2(2q_1 + q_2)$$

Now,

$$F_3(p_1, Q_1, p_2, Q_2) = F_1(q_1, q_2, Q_1, Q_2) - q_1\frac{\partial F_1}{\partial q_1} - q_2\frac{\partial F_2}{\partial q_2} \quad \because p_j = \frac{\partial F_1}{\partial q_j} \;\; j = 1, 2$$
$$F_3(p_1, Q_1, p_2, Q_2) = 2q_1p_2 + q_2p_2 - q_1p_1 - q_2p_2$$
$$F_3(p_1, Q_1, p_2, Q_2) = 2q_1p_2 - q_1p_1$$
$$F_3(p_1, Q_1, p_2, Q_2) = q_1(2p_2 - p_1)$$

and $$F_3(p_1, Q_1, p_2, Q_2) = Q_1(2p_2 - p_1)$$

Problem: 7.10- Find under what condition $Q = \frac{\alpha p}{x}, P = \beta x^2$, where $\alpha \& \beta$ are constants, represents a canonical transformation for a system of one degree of freedom and obtain a suitable generating function. Apply the transformation to the solution of linear harmonic oscillator.

Solution

If

$$Q = \frac{\alpha p}{x} \tag{7.60}$$

and

$$P = \beta x^2 \tag{7.61}$$

is canonical, then $pdq - PdQ$ will be total differential or $[Q, P] = 1$. So from Eq.(7.60), we get

$$\frac{\partial Q}{\partial x} = -\frac{\alpha p}{x^2}$$

and,

$$\frac{\partial Q}{\partial p} = -\frac{\alpha}{x}$$

And from Eq.(7.61), we get

$$\frac{\partial P}{\partial x} = 2\beta x$$

$$\frac{\partial P}{\partial p} = 0$$

Thus,

$$\begin{aligned}
[Q, P] &= \frac{\partial Q}{\partial x}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial x} \\
1 &= -\frac{\alpha p}{x^2}(0) - \frac{\alpha}{x}(2\beta x) \qquad \because [Q, P] = 1 \\
1 &= 0 - 2\alpha\beta \\
1 &= -2\alpha\beta \\
\text{or} \quad \alpha &= -\frac{1}{2\beta}
\end{aligned} \tag{7.62}$$

Eq.(7.62) gives the required condition. Let $\beta = 1$, Eq.(7.62) implies $\alpha = -\frac{1}{2}$, so the transformation equations are $Q = -\frac{p}{2x}$ and $P = x^2$. Now,

$$\begin{aligned}
pdx - PdQ &= pdx - x^2\left[-\frac{1}{2x}dp + \left(-\frac{p}{2}\right)\left(-\frac{1}{x^2}\right)dx\right] \\
pdx - PdQ &= pdx - x^2\left[-\frac{1}{2x}dp + \frac{p}{2x^2}dx\right] \\
pdx - PdQ &= pdx + \frac{x}{2}dp - \frac{p}{2}dx \\
\text{or} \quad pdx - PdQ &= pdx + xdp - \frac{x}{2}dp - \frac{p}{2}dx \\
pdx - PdQ &= d(xp) - \frac{1}{2}d(xp) \\
pdx - PdQ &= d\left[xp - \frac{1}{2}xp\right] \\
pdx - PdQ &= d\left[\frac{1}{2}xp\right] \\
pdx - PdQ &= dF_1
\end{aligned}$$

Now the generating function is;

$$dF_1 = d\left[\frac{1}{2}xp\right]$$

$$\begin{aligned}
\Longrightarrow \quad F_1(x,Q) &= \frac{1}{2}xp \\
F_1 &= \frac{1}{2}x\left[\frac{Qx}{\alpha}\right] && \because \text{Using Eq.(7.60)} \\
F_1 &= \frac{1}{2}Q\frac{x^2}{\alpha} \\
F_1 &= \frac{1}{2}Qx^2\frac{1}{-1/2} && \because \alpha = -1/2 \\
F_1 &= -Qx^2
\end{aligned}$$

is the required generating function of first kind.

Problem: 7.11- Show that the direct transformation condition for canonical are given immediately by the symplectic condition expressed in the form $JM = \widetilde{M}^{-1}J$.

Solution

$$JM = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial P}{\partial p} \end{bmatrix}$$

$$JM = \begin{bmatrix} \frac{\partial P}{\partial q} & \frac{\partial P}{\partial p} \\ -\frac{\partial Q}{\partial q} & -\frac{\partial Q}{\partial p} \end{bmatrix} \tag{7.63}$$

$$\widetilde{M} = \begin{bmatrix} \frac{\partial Q}{\partial q} & \frac{\partial P}{\partial q} \\ \frac{\partial Q}{\partial p} & \frac{\partial P}{\partial p} \end{bmatrix} \tag{7.64}$$

Now,

$$\widetilde{M}^{-1}J = \begin{bmatrix} \frac{\partial P}{\partial p} & -\frac{\partial P}{\partial q} \\ -\frac{\partial Q}{\partial p} & -\frac{\partial Q}{\partial q} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$\widetilde{M}^{-1}J = \begin{bmatrix} \frac{\partial P}{\partial q} & \frac{\partial P}{\partial p} \\ -\frac{\partial Q}{\partial q} & -\frac{\partial Q}{\partial p} \end{bmatrix} \tag{7.65}$$

From Eqs.(7.63) and (7.65), it is clear that

$$JM = \widetilde{M}^{-1}J$$


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